Journal of Economic Behavior and Organization 181 (2021) 169–210

Contents lists available at ScienceDirect

Journal of Economic Behavior and Organization

journal homepage: www.elsevier.com/locate/jebo

Dynamic R&D competition under uncertainty and strategic disclosure☆

Yangbo Song[a], Mofei Zhao[b,*]

[a] The School of Management and Economics, The Chinese University of Hong Kong (Shenzhen), Shenzhen, China
[b] School of Economics and Management, Beihang University, Beijing, China

ARTICLE INFO

Article history:
Received 18 June 2018
Revised 28 November 2020
Accepted 28 November 2020

JEL classification:
D83
L13
O31,

Keywords:
R&D competition
Uncertainty
Strategic disclosure

ABSTRACT

We study a dynamic two-stage R&D competition with unknown difficulty of the first stage and a reward for declaring the success in each stage. A competing firm can choose whether and when to disclose the solution to the first stage, as well as whether and when to quit. We characterize the unique equilibrium for homogeneous firms, which always exhibits a disclose-withhold-exit pattern as time evolves. In terms of social welfare, a competition is not always optimal: when research costs are high, it is socially more desirable to assign the project to a single firm. When firms are heterogeneous, a cost advantage always leads to an information advantage while a research ability advantage may generate opposite outcomes.

1. Introduction

In the 1980s, two research teams – one composed of Lockheed, Boeing and General Dynamics and the other composed of Northrop and McDonnell Douglas – stood out as finalists in the Advanced Tactical Fighter competition launched by the U.S. Air Force. In the research process, the aim of the first and highly uncertain stage is to develop a new supersonic engine. If a team devotes considerable time to the stage in vain, it realizes that this problem is likely more difficult than expected and may reconsider whether it is worthwhile to remain in the competition. On the other hand, after inventing the new engine, a team must decide between immediately marketing the engine and keeping it a secret for a period of time. The former option yields an immediate profit but enables the opponent to catch up in research by purchasing the engine, while the latter option protects the leading position in the race at the cost of possibly losing part of the current profit. A participant's optimal behavior in this type of strategic environment then involves answering two important questions: when should it make its key invention public, and when should it exit the race without any good news?

In this paper, we propose a model to comprehensively analyze firms' strategic decisions in a dynamic R&D competition under uncertainty and answer the above questions. As can be gleaned from the above example, our model focuses on two

☆ We are grateful to Ichiro Obara, Moritz Meyer-ter-Vehn, Qianfeng Tang, Siwei Cao, and the seminar audience at Nanjing Audit University, Shanghai University of Finance and Economics, and Xiamen University for helpful suggestions. This work was partially funded by Projects 71703137 and 71803167 supported by NSFC.

* Corresponding author.
E-mail addresses: yangbosong@cuhk.edu.cn (Y. Song), 10708@buaa.edu.cn (M. Zhao).

https://doi.org/10.1016/j.jebo.2020.11.033
0167-2681/

stylized features of R&D competitions. First, an early but important stage often exhibits uncertainty over the likelihood of success, which has to be resolved via learning over time. Second, upon success in the early stage, a firm can strategically control technology or information spillovers and hence partially manipulate its opponent's belief, at the expense of its immediate profit. We embed these features in a standard Bayesian game framework.

We consider two firms competing on a continuous timeline for a research project consisting of two stages, 1 and 2, with stage 1 being an innovative stage to develop an intermediate product and stage 2 being a commercial stage to release a final product. It is necessary to solve stage 1 before work can begin on stage 2. The success in each stage arrives at a Poisson rate, which is fixed for stage 2 but uncertain for stage 1: this rate is either positive (solvable) or zero (unsolvable). Whenever a firm solves stage 1, it can either disclose or withhold the solution. Disclosure entails a technology spillover such that both firms can begin working on the next stage. Each stage has its own value, which is entirely captured by the firm that discloses the solution if its opponent has not yet solved the stage and is shared if both firms have the solution by the time either firm discloses it.[1] Working on the innovative stage 1 involves reallocation of current resource and possibly acquiring external resource, and bears a positive cost per unit time; in contrast, the commercial stage 2 is assumed to have zero cost as it is part of a firm's routine business operation. Finally, a firm is free to exit the competition once and for all at any moment.

We first characterize the unique equilibrium of this game, which is symmetric and always has a "disclose-withhold-exit" pattern. There are two important time thresholds: a firm will disclose its success on stage 1 immediately at any time instant before the first threshold, withhold the solution if it succeeds between the thresholds, and exit if it has not succeeded by the second threshold (and stay if it has). In other words, technological spillover does not always take place even when it is very rewarding: as long as the firms are naturally allowed to conceal their success and to exit the competition at will, invention of the intermediate product that occurs after the first time threshold will never affect the rival's payoff or incentive. This unique equilibrium behavioral pattern is invariant if we assume alternatively that disclosure only conveys the information about the success, rather than the actual solution, to the opponent.

The reasoning behind our result is a firm's learning dynamics. If the rival has remained in the competition long enough but has not received any good news from either firm, the rival should quit because information updating implies that stage 1 is likely to be unsolvable. In light of this, when a firm solves stage 1 at a time close to the exit point, it will withhold the solution, hoping that the rival will exit soon enough. Nevertheless, if the solution comes early, there is less benefit and more risk from withholding it and waiting, and thus, the firm will just disclose the solution to guarantee the one bird in hand.

The outlook of the whole project's completion depends on the market structure, measured by the number of competing firms, as well as each firm's strategic behavior under the particular structure. When the success of the entire project is crucial to the society, it is important for the relevant authorities (the Ministry of Defense, for instance) to examine this comparison to select the socially desirable structure. Intuitively, the social welfare is positively related to the probability of completing the entire project. If disclosure were automatic, this probability would have a one-to-one relationship with the equilibrium total time spent by all firms without solving stage 1, which is a constant regardless of the number of firms. However, with strategic disclosure, the equilibrium property that each firm will always withhold the solution when time approaches the opponent's exit point naturally raises the following question: is the project more likely to succeed, and is the social welfare always higher, under a two-firm competition than a one-firm monopoly? By explicitly characterizing a social welfare function, we find that competition is not always desirable. In particular, we show that when the research cost is relatively high, the ex ante probability of completing the entire project, as well as the social welfare, is higher when only one firm works on it than when two firms compete. The underlying intuition is that when the firms still hold high beliefs but also face a considerable cost of remaining in the competition, the fear of falling behind in stage 1 will outweigh that of stage 1 being unsolvable, causing them to exit prematurely compared to the case of a single firm. As an implication for policy, if the success of an R&D project is highly valuable but the likelihood of success is uncertain, the relevant authorities should carefully assess the relative magnitude of the difficulty of the problem and its associated cost and may consider privately assigning the research task to one firm rather than tendering it to multiple competing firms.

Our basic model studies homogeneous firms and the resulting symmetric equilibrium, but rivals are not similar to one another in all real business competitions. We analyze two typical cases of asymmetric competition: different costs and different research abilities. Although these are both characteristic measures of a firm's strength, we find that a lower cost and high ability do not have the same implications for equilibrium behavior. An advantage in cost always leads to an advantage in information, in the sense that the cost-efficient firm begins withholding its success in stage 1 earlier than its opponent and exits the game later than its opponent. However, the same pattern does not occur under asymmetric research abilities. The firm that can solve stage 1 faster may either exit earlier or later than its opponent, depending on the parameter specification.

The remainder of this paper is organized as follows: Section 2 reviews the related literature. Section 3 presents the model. Section 4 characterizes the equilibrium and discusses its economic implications. Section 5 analyzes asymmetric competitions. Section 6 concludes.

[1] This value can be viewed as potential profits from patenting or marketing the product of the corresponding stage.

2. Literature review

Early economic models of R&D races explore the investment decisions that firms make in an effort to reduce production cost (Dasgupta and Stiglitz, 1980a; 1980b), obtain a rewarding technological breakthrough (Reinganum, 1981), or secure intellectual property rights (Loury, 1979; Lee and Wilde, 1980). Among these seminal works, Loury (1979) proposes a framework with exponentially distributed time to innovation success, which has become one of the standard ways to depict an R&D competition in subsequent research. We adopt this approach in our model with an added structure of uncertainty. While the early works usually emphasize a firm's optimal investment problem in a one-shot game, we focus on the optimal delay or even concealment of intermediate innovation in a two-stage competition with continuous time.

In the domain of R&D races with more than one stage, a number of theoretical studies have attempted to depict firms' strategic interaction at an intermediate stage. Grossman and Shapiro (1987) and Harris and Vickers (1987) extend Loury's model to two stages and study the competing firms' investment incentives. They find that competition is most intense when firms are close to one another in the race, in the sense that they are most willing to increase their research efforts. Subsequent studies such as Bloch and Markowitz (1996) focus on disclosure delay of intermediate results, as we do, but base their analysis on complete information in the research process and automatic announcement of success in the intermediate stage. The general conclusion drawn from this literature is that success in intermediate stages is beneficial to both the inventor and its opponent. De Fraja (1993) comes to a similar conclusion by modeling the spillover as an investment that will benefit all competing firms.

Choi (1991) is the first to study uncertainty in success rates, whose model features a two-stage race in which the success rate of the first stage is unknown, and the result accords with the above in that one firm's success in the early stage always benefits the opponent by signaling a low difficulty level. This proposition, nevertheless, stems from the assumption that no firm can conceal its success, i.e., a firm automatically knows when its opponent has solved the first stage. Several related works have raised a discussion around the idea of concealing innovation for an advantage in the subsequent race. Scotchmer and Green (1990) study a two-stage patent race under complete information and allow firms to conceal their intermediate innovation. They predict that firms either never withhold or always withhold a success, depending on parameter values. Hendricks (1992) introduces incomplete information in types of firms, and shows that firms strategically delays adoption of novel technologies to build a reputation as imitators. Bag and Dasgupta (1995) studies a variation of Choi's model without technological spillover, in which the research capability of one of the competing firms is unknown to all firms, and they attribute the incentive of disclosing intermediate success to signaling own strength. In comparison to these works, disclosure in our model bears both direct effects (the immediate reward and technological spillover) as well as indirect informational effects (informing the opponent about difficulty of early stage), which not only reflect a natural trade-off in the choice of when to disclose or conceal a success, but also lead to interesting results on further topics such as welfare and equilibrium behavior under firm heterogeneity. Other approaches for modeling payoff-related uncertainty in R&D competitions can be found in, for instance, Malueg and Tsutsui (1996), Chatterjee and Evans (2004) and Moscarini and Squintani (2010). A recent development in this field is Bobtcheff et al. (2017). They also focus on two-player priority races in which the solution to a valuable problem is privately observed and every player with the solution needs to decide when to disclose her result. The longer a player waits, the larger the value of her solution becomes if she still preempts her opponent. As a result, players in their model behave in the opposite way to ours: they withhold the solution at the beginning and only disclose it when it is "mature", i.e., the value has grown considerably after a sufficiently long time.

In terms of modelling techniques, our paper relates to the growing literature on learning through experimentation. One leading approach in this line of research is the strategic bandit model (Keller et al., 2005; Keller and Rady, 2010; Heidhues et al., 2015; Das et al., 2020), where a player learns about the true payoff from a risky arm by observing the actions and resulting outcomes of all players. Our model also depicts experimentation in continuous time, but with private learning and strategic disclosure of learned information. From the more specific perspective of R&D, choice over a safe research method and a risky one has been studied under both collaboration (Dong, 2018; Guo and Roesler, 2018) and competition (Akcigit and Liu, 2016; Das and Klein, 2020). In these works, a firm makes decision by weighing its privately learned expected payoff from the risky path against a commonly known outside option. A firm in our framework faces a similar problem, plus the additional concern of how its choice in the current stage may affect the outlook of competition in the next. Horner and Skrzypacz (2017) provides a comprehensive survey of this field.

There is also a large literature in law and economics analyzing information disclosure in patent races in a rather different framework. As indicated by representative studies such as Denicolo and Franzoni (2004), Kultti et al. (2007) and Hopenhayn and Squintani (2016), the trade-off between secrecy and patenting is crucial for understanding equilibrium behavior and for optimal patent design. However, in this line of research, a firm does not strategically disclose information that is useful to its opponent to secure the immediate benefit. Instead, incentives for information disclosure include signaling strength and commitment to the race (Anton and Yao, 2003; Gill, 2008), inducing the exit of risk-averse competitors (Bhattacharya and Ritter, 1983; Lichtman et al., 2000; Baker and Mezzetti, 2005) or establishing prior art as a defensive measure (Parchomovsky, 2000).

3. Model

3.1. The R&D competition game

Two-Stage R&D. Consider two firms, i and j, that compete to successfully complete a research project on a continuous timeline. Each firm has a research department and a commercial department. To complete the project, a firm has to complete two consecutive stages: an innovative stage by the research department to invent a new intermediate product (such as a supersonic engine for a fighter jet or a high-performance graphics processing unit for a graphics card) and a commercial stage by the commercial department to develop a marketable final product. Success in stage 1 is required for any work on stage 2. Once stage 2 is solved by either firm, the game ends. For each firm in either stage, it can decide at any time instant whether to continue research or to exit the competition, but this decision is not observable by the opponent. The only publicly observable action is whether either firm has disclosed its success in stage 1, the consequence of which will be specified in detail below.

Technology and Uncertainty. At $t = 0$, the two research departments begin developing stage 1. The firms face the same Poisson rate of success equal to $\lambda \in \{H > 0, 0\}$. λ measures the intrinsic difficulty of stage 1. In other words, each firm can either solve stage 1 with positive probability, or it can never succeed. The value of λ cannot be observed by either firm but has to be learned over time. We use $\tilde{\lambda}^i(t)$ to denote firm i's belief (resp. $\tilde{\lambda}^j(t)$ for firm j's belief), i.e., the probability that $\lambda = H$ at time t from firm i's perspective. For simplicity, we assume that at $t = 0$, both firms hold an identical prior $\tilde{\lambda}^i(0) = \tilde{\lambda}^j(0) = \alpha \in (0, 1)$.

Admittedly, a more realistic measure of uncertain difficulty in research would be a λ which presumably increases in time when stage 1 is solvable, due to the firm's accumulating knowledge. Nevertheless, as will be implied by the belief-updating process in the next section, the firms' beliefs may be non-monotone in this setting, rendering the analysis intractable. Hence we choose to adopt the above simpler configuration for illustrating the main economics of the model, and leave the alternative framework for future investigation.

Strategic Disclosure. Once a firm solves stage 1, it can choose whether to disclose or to withhold its invention. Disclosure can only occur once. When a firm discloses the solution, if its opponent has not succeeded yet, the firm claims all credit and receives a reward of $p_1 > 0$. However, if both firms have solved stage 1 by the time of disclosure or when the game ends without such disclosure, the firms will ultimately receive $\frac{p_1}{2}$ each.[2] Once the solution to stage 1 is disclosed by either firm, the product becomes available to both firms, who then begin working on stage 2, the second and final product, with an i.i.d. Poisson rate of success $\mu > 0$. We assume that there is no uncertainty in success rate for the more routine commercial stage, i.e. μ is common knowledge, thereby allowing us to focus on uncertainty in stage 1 without considering insignificant technical details. If a firm chooses not to disclose the solution after solving stage 1, it can still work on stage 2 on its own. Whichever firm solves stage 2 first receives a reward of $p_2 > 0$.

Therefore, a firm's payoff from the game before cost and discount is

$$\begin{cases} p_1 + p_2 & \text{if it solves stage 2 before the opponent solves stage 1} \\ \frac{p_1}{2} + p_2 & \text{if it solves but withholds stage 1, and solves stage 2 first while the opponent} \\ & \text{has also solved stage 1} \\ p_1 & \text{if it solves and discloses stage 1 before the opponent does, but the opponent} \\ & \text{solves stage 2 first} \\ \frac{p_1}{2} & \text{if it solves but withholds stage 1, but the opponent solves stage 2 first} \\ 0 & \text{if the opponent solves stage 2 before it solves stage 1} \end{cases}$$

Cost and Discount. Remaining in the competition may be costly. We assume that during stage 1, each firm pays a cost per unit of time $c > 0$, which can be interpreted as the opportunity cost of the resources needed to establish a new department, hire external experts, etc. However, the commercial stage is part of the firm's regular business and does not require reallocation of resources, so we assume that the cost is zero for stage 2. We will briefly discuss the case with positive stage-2 cost in Section 4.1.

The firms discount their rewards and costs exponentially with a common rate $r \geq 0$.

Exit and Re-Entry. The research department can terminate its research on stage 1, which is a once-and-for-all decision. However, if the opponent firm has disclosed the solution to stage 1, the commercial department can return to compete for stage 2. As mentioned before, exit is not observed by the opponent.

3.2. Strategy, belief and equilibrium

Strategy. Before solving stage 1, a firm is only free to choose when to exit. After finishing the intermediate product, it then conditions its decision of disclosing/withholding the solution on the history. The relation among players' information, strategies, beliefs and possible actions can be depicted by Fig. 1 below. The left-hand side indicates a firm's primitive

[2] In many applications of the model, this can be regarded as a common scenario in which the firms file lawsuit for the ownership of the invention. In principle, whether to file the lawsuit and obtain half of the reward is a choice, but here it must be the optimal action to take. Hence, without loss of generality, we assume that this process is automatic. In Section 4.3, we will discuss the effect of alternative policy rules on equilibrium characterization.

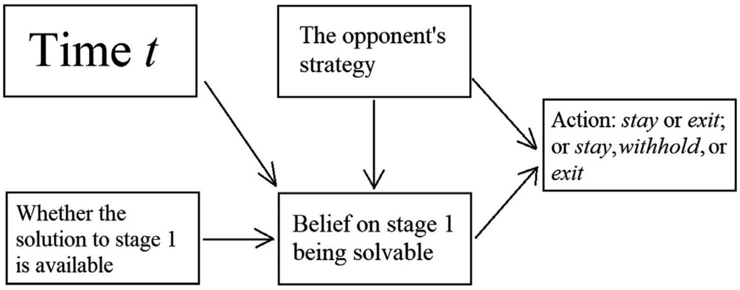

Fig. 1. A firm's decision process.

information: time and state (whether stage 1 is known to be solvable). These two pieces of information, together with the firm's perception of its opponent's strategy, uniquely determine the firm's belief on λ. Given this belief, the firm then selects its optimal action in response to the opponent's strategy.

There are two natural approaches of defining a strategy in our game. The first is to define a strategy as a mapping directly from belief to action. However, to determine whether, when and how a firm's belief may reach a certain value (in particular the value that makes the firm, without solving stage 1, indifferent between staying in the game and quitting), the primitive information must also be involved. In addition, the calculation of beliefs will inevitably take into account the opponent's strategy which is endogenous. Therefore we find it clearer and more concise to adopt the second approach, i.e. to define a strategy as a mapping from the primitive information to action. The two approaches are in fact equivalent: given the opponent's strategy, a firm's primitive information maps to one and only one value of its belief.

Specifically, letting $T = [0, \infty)$ be the set of time, firm i's *strategy* $\sigma^i = (\sigma_1^i, \sigma_2^i)$ consists of two mappings: $\sigma_1^i : T \to \{Stay, Exit\}$ when neither the firm has solved stage 1 nor has its opponent disclosed the solution; $\sigma_2^i : T \to \{Disclose, Withhold\}$ when the firm has solved stage 1 but neither firm has disclosed the solution. We have omitted the possibility of a firm exiting after it has solved stage 1, since exiting then is never optimal. We assume that a firm uses Bayesian updating whenever possible.

We focus on strategies that are piecewise continuous and history independent. That is, a *strategy profile* for firm i can be characterized by a number of cutoffs $t_1^i, t_2^i ... t_n^i$ in the following way: for an arbitrary k, the firm's action at time t is the same for every $t \in [t_k^i, t_{k+1}^i)$ (we use left-closed and right-open intervals for consistent notations, without loss of generality). We call $[t_k^i, t_{k+1}^i)$ a "disclose region" if the firm immediately discloses its success in stage 1 (if any) in $[t_k^i, t_{k+1}^i)$; we call $[t_k^i, t_{k+1}^i)$ a "withhold region" if the firm withholds its success in $[t_k^i, t_{k+1}^i)$; and we call $[t_k^i, t_{k+1}^i)$ an "exit region" if the firm exits the game at every $t \in [t_k^i, t_{k+1}^i)$ in the absence of success.

Belief. The key difference made by this model is twofold: the existence of uncertainty and the firms' strategic response to it. Firms do not directly observe λ and are free to conceal their success in stage 1 to partly manipulate the opponent's belief. As mentioned above, each firm holds a belief at time t regarding the difficulty of the project. A firm updates its belief through (1) its own progress (success or not) in stage 1 and (2) its opponent's disclosed success or silence. It is important to note here that the belief-updating process of each firm is affected by its opponent's strategy.

The trajectory of $\tilde{\lambda}^i(t)$ is simple if t lies in (one of) firm j's disclose region(s). In the absence of a disclosure of success by the opponent or a success by the firm itself, $\tilde{\lambda}^i(t)$ evolves as follows:

$$\tilde{\lambda}^i(t) = \frac{\alpha e^{-2Ht}}{\alpha e^{-2Ht} + 1 - \alpha}. \tag{1}$$

However, whenever either firm succeeds in this region, both firms' beliefs jump to 1 because the successful firm will immediately disclose the solution.

In (one of) firm j's withhold region(s), the updating process is somewhat more complex, as silence does not explicitly suggest whether the opponent has solved stage 1; instead, it implies that the opponent has not solved stage 2, thus implicitly undermining the likelihood of its success in stage 1. When t lies in a withhold region $[t_k^j, t_{k+1}^j)$, in the absence of a success by firm i itself, $\tilde{\lambda}^i(t)$ evolves as follows:

$$\tilde{\lambda}^i(t) = \frac{\alpha e^{-2Ht_k^j - H(t-t_k^j)}\left(e^{-H(t-t_k^j)} + \int_0^{t-t_k^j} He^{-sH} e^{-(t-t_k^j-s)\mu} ds\right)}{\alpha e^{-2Ht_k^j - H(t-t_k^j)}\left(e^{-H(t-t_k^j)} + \int_0^{t-t_k^j} He^{-sH} e^{-(t-t_k^j-s)\mu} ds\right) + 1 - \alpha}. \tag{2}$$

Whenever a firm succeeds in this region, only its own belief jumps to 1 since it will keep the good news private.

Intuitively, $\tilde{\lambda}^i(t)$ should be decreasing, as waiting in vain for success can only indicate that stage 1 is increasingly likely to be unsolvable. This is confirmed by the following lemma.

Lemma 1. *$\tilde{\lambda}^i(t)$ always decreases in t.*

Proof. For (1), observe that e^{-2Ht} decreases in t; thus, $\tilde{\lambda}^i(t)$ decreases in t.

For (2), when $\mu \neq H$ we have

$$e^{-H(t-t_k^j)} + \int_0^{t-t_k^j} He^{-sH} e^{-(t-t_k^j-s)\mu} ds = e^{-H(t-t_k^j)} + \frac{H}{\mu - H}(e^{-(t-t_k^j)H} - e^{-(t-t_k^j)\mu})$$
$$= \frac{\mu}{\mu - H} e^{-H(t-t_k^j)} - \frac{H}{\mu - H} e^{-(t-t_k^j)\mu}.$$

The derivative of the right-hand side with respect to t is equal to

$$-\frac{\mu H}{\mu - H}(e^{-H(t-t_k^j)} - e^{-\mu(t-t_k^j)}),$$

which is always negative. Hence, we can conclude that the numerator of (2) decreases in t and thus $\tilde{\lambda}(t)$ decreases in t.

When $\mu = H$, we have

$$e^{-H(t-t_k^j)} + \int_0^{t-t_k^j} He^{-sH} e^{-(t-t_k^j-s)\mu} ds = e^{-H(t-t_k^j)} + H(t - t_k^j) e^{-H(t-t_k^j)},$$

which again decreases in t. Therefore, $\tilde{\lambda}^i(t)$ decreases in t. □

The reasoning behind Lemma 1 is straightforward yet useful. In the disclose region, in the absence of disclosure, a firm's belief update is twofold. First, its unsuccessful research decreases its estimation of λ; second, the implied public information that its opponent has also not succeeded again decreases its estimation of λ. In the withhold region, although the firm no longer observes its opponent's progress, it knows at least that the opponent has not completed stage 2 (otherwise, the opponent would disclose everything and win the game) and thus is also less likely to have completed stage 1. Hence, we can conclude that $\tilde{\lambda}^i(t)$ always decreases in t, with updating in a disclose region being faster than in a withhold region.

Equilibrium. Now we define the equilibrium of this game. Following the conventional characterization, a *perfect Bayesian equilibrium*, or simply *equilibrium*, consists of a strategy profile (σ^1, σ^2) (which can also be characterized as $\{t_k^i\}_{k=1,\cdots,n}^{i=1,2}$) and a belief updating rule $(\tilde{\lambda}^1(t), \tilde{\lambda}^2(t))$, such that (1) for each firm, at any time $t \geq 0$, following the action specified in the strategy profile maximizes the firm's expected continuation payoff given its belief and the opponent's strategy; (2) the belief updating rule is correct, i.e. it satisfies (1) and (2) above. An immediate property of every equilibrium is given by Lemma 2 below.

Lemma 2. *In every equilibrium, there exists $\bar{t}$ such that a firm exits at every $t \in [\bar{t}, \infty)$ if it has not solved stage 1.*

Proof. As $t \to \infty$, in the absence of success, $\tilde{\lambda}^i(t) < \frac{\alpha e^{-Ht}}{\alpha e^{-Ht} + 1 - \alpha} \to 0$. Hence, as $\tilde{\lambda}^i(t) > 0$, we have $\tilde{\lambda}^i(t) \to 0$. Then, we compare the firm's incentives for remaining and exiting. The incentive to remain for an additional time increment dt is less than $\tilde{\lambda}^i(t)H(p_1 + p_2)dt$, which approaches $o(dt)$ as $t \to \infty$. By our assumption of a fixed $c > 0$, an exit point always exists. □

The intuition for the existence of an exit region in every equilibrium is that no firm is willing to persist indefinitely in the competition when remaining is costly and stage 1 may just be unsolvable. As time passes with no good news arriving, a firm believes that $\lambda = 0$ is increasingly likely and will ultimately decide to exit. Recall that every strategy profile can be characterized by a number of cutoffs $t_1^i, t_2^i ... t_n^i$; Lemma 2 indicates that $[t_n^i, \infty)$ must be the only exit region in every equilibrium.

3.3. Assumptions on parameters

In this section, we specify the following three assumptions on the set of parameters $\{\alpha, H, \mu, p_1, p_2, c, r\}$ that are made throughout the paper. We will be more specific about the use of each assumption in the subsequent sections.

A1. $\alpha H(p_1 + p_2) > c$. This assumption guarantees that at least at $t = 0$ each firm has a positive rate of expected net return.

A2. This assumption makes it possible for both disclosure and withholding to occur in equilibrium, and consists of two conditions.

First, r is sufficiently small; in particular, $p_1 + \frac{\mu}{2\mu + r} p_2 < \frac{\mu}{\mu + r}(p_1 + p_2)$. This condition is always satisfied when $r = 0$. The left-hand side is a firm's payoff in competition, following immediate disclosure which secures it p_1; the right-hand side is its payoff without competition from withholding the solution to stage 1 until it solves stage 2. If this condition were violated, each firm would always prefer disclosure.

Second, the first stage is sufficiently valuable; in particular, $p_1 H > p_2 \mu$. This condition makes it possible for disclosure to occur in equilibrium even when there is no discount. If this condition were violated, our main results still hold, but in equilibrium no firm will disclose stage 1's solution, namely both firms will voluntarily withhold the solution until either of them succeeds in stage 2. In various industries, a rather valuable intermediate product indeed constitutes the main source of profit for its producer. Notable examples include the majority of OEM markets, the market for graphics card and Bitcoin miner, TV and monitor, pharmaceutical industry, and in some cases academic research.

A3. This assumption guarantees the existence of an equilibrium. We assume that either of the following two conditions is satisfied:

$$\begin{cases} \alpha \leq \frac{1}{2} \\ c \leq \frac{H(p_1+p_2)\left(\frac{-\mu^2}{(H-\mu)(\mu+H+r)}e^{-H\nabla}+\frac{\mu H}{(H-\mu)(2\mu+r)}e^{-\mu\nabla}\right)}{e^{H\nabla}+\frac{H}{H-\mu}e^{-\mu\nabla}-\frac{\mu}{H-\mu}e^{-H\nabla}}, \end{cases}$$

where $\nabla = \frac{\ln(\frac{\frac{p_1+\frac{\mu}{2\mu+r}p_2}{p_1+p_2}-B}{A})}{-(H+\mu+r)} > 0$, with $A = \frac{H\mu^2}{(\mu+r)(H+\mu+r)(2\mu+r)}$ and $B = \frac{2\mu^2+\mu r+H\mu}{(H+\mu+r)(2\mu+r)}$. When $r = 0$, the second inequality above is simplified as follows:

$$c \leq \frac{H\frac{p_1+p_2}{2}\left(\frac{\mu}{H+\mu}e^{-H\nabla}+\frac{H}{H-\mu}e^{-\mu\nabla}-\frac{\mu}{H-\mu}e^{-H\nabla}\right)}{e^{H\nabla}+\frac{H}{H-\mu}e^{-\mu\nabla}-\frac{\mu}{H-\mu}e^{-H\nabla}},$$

where $\nabla = \frac{\ln\frac{H(p_1+p_2)}{p_1H-p_2\mu}}{H+\mu} > 0$.

A3 is a sufficient condition for our subsequent results. It is equivalent to the following statement: in every equilibrium, whenever a firm enters a withhold region, its belief on λ is at most $\frac{1}{2}$. Intuitively, this condition is satisfied when the firms begin with a prior $\alpha \leq \frac{1}{2}$. Alternatively, when c is small so that each firm will only exit given a low belief, they will only enter a withhold region given a low belief as well because withholding stems from the hope for the opponent to exit in the near future.

With such beliefs, we obtain monotone incentives for a firm in its withhold region. As time evolves, in absence of any breakthrough or disclosure, a firm becomes more pessimistic as its own failure and the opponent's silence imply that stage 1 is more likely to be impossible; on the other hand, it becomes more optimistic as the opponent's silence may imply that it is going to exit soon. It turns out that with a belief of at most $\frac{1}{2}$, a firm always becomes less willing to stay as time evolves, i.e. the first incentive dominates the second, which in turn guarantees the existence of an equilibrium.

4. Equilibrium characterization and economic implications

4.1. Unique equilibrium with no discount

Our first main result is the explicit characterization of the unique equilibrium when $r = 0$. We begin by characterizing each firm's behavior in every possible equilibrium.

Lemma 3. *In every equilibrium, the behavior of firm i (resp. firm j) exhibits a disclose-withhold-exit pattern: there exist cutoffs t_1^i, t_2^i such that at time t, the firm*

$$\begin{cases} \text{discloses the solution to stage 1 if } t \in [0, t_1^i) \\ \text{withholds the solution to stage 1 if } t \in [t_1^i, t_2^i) \\ \text{exits if } t \in [t_2^i, \infty) \text{ when it has not solved stage 1.} \end{cases}$$

Proof. See Appendix A. □

When time approaches the opponent's exit point, a firm's optimal action – if it has solved stage 1 – must be withholding the solution. By doing so, it will capture the entire benefit $p_1 + p_2$ once the opponent exits, while disclosure may save the opponent. Furthermore, we can show that once a firm is indifferent between disclosing and withholding the solution at the beginning of this final withhold region, it cannot prefer withholding the solution for any time length before that. Hence, there can be at most one disclose region.

Now, we are ready to present our main theorem.

Theorem 1. *The R&D competition game has a unique equilibrium. The equilibrium is a symmetric disclose-withhold-exit strategy profile characterized by cutoffs $t_1, t_2 \geq 0$ such that $t_1^i = t_1^j = t_1$ and $t_2^i = t_2^j = t_2$ and that at time t, each firm*

$$\begin{cases} \text{discloses the solution to stage 1 if } t \in [0, t_1) \\ \text{withholds the solution to stage 1 if } t \in [t_1, t_2) \\ \text{exits if } t \in [t_2, \infty) \text{ when it has not solved stage 1.} \end{cases}$$

Proof. See Appendix B. □

To sketch the proof, we first show that the only possible equilibrium strategy profile must be symmetric: otherwise, the firms must be facing different costs, which is a contradiction. Henceforth, we can omit the superscripts i, j and refer to the time cutoffs as t_1 and t_2. Then, we explicitly characterize these cutoffs. This is achieved by combining two necessary conditions for an equilibrium: the indifference condition, whereby a firm obtains the same expected payoff from disclosure and from withholding the solution at t_1 if $t_1 > 0$, and the breakeven condition that its instantaneous rate of payoff is equal to the cost per unit of time at t_2. The conditions are as follows:

$$p_1 + \frac{p_2}{2} \le e^{-H(t_2-t_1)} e^{-\mu(t_2-t_1)}(p_1+p_2) + \int_{t_1}^{t_2} \mu e^{-\mu(s-t_1)} e^{-H(s-t_1)} ds (p_1+p_2)$$
$$+ \int_{t_1}^{t_2} H e^{-H(s-t_1)} e^{-\mu(s-t_1)} ds \frac{(p_1+p_2)}{2}, \text{ “=” if } t_1 > 0$$
$$c = \tilde{\lambda}(t_2) H \frac{(p_1+p_2)e^{-H(t_2-t_1)} + \frac{(p_1+p_2)}{2}\int_0^{t_2-t_1} He^{-Hs}e^{-(t_2-t_1-s)\mu}ds}{e^{-H(t_2-t_1)} + \int_0^{t_2-t_1} He^{-Hs}e^{-(t_2-t_1-s)\mu}ds}.$$

If a firm discloses the solution before the opponent enters a withhold region, it is certain that the opponent has not solved stage 1, which guarantees the firm a payoff of p_1. In addition, From the moment of disclosure the two firms compete for stage 2 on equal ground and each earns $\frac{p_2}{2}$ in expectation. Hence, the payoff from disclosure at t_1, i.e. the left-hand side of the first condition, is $p_1 + \frac{p_2}{2}$.

Withholding the solution from t_1 onwards, on the other hand, may result in different payoffs. If neither the firm nor its opponent achieves a success between t_1 and t_2 (solving stage 2 for the firm, stage 1 for its opponent) or the firm solves stage 2 before its opponent solves stage 1, the firm has the whole reward $p_1 + p_2$. Otherwise, if the opponent solves stage 1 before the firm solves stage 2, they will ultimately share p_1 and once again compete fairly for stage 2, and each firm's expected payoff becomes $\frac{p_1+p_2}{2}$. The right-hand side of the first condition is the expectation of these payoffs.

The right-hand side of the second condition represents a firm's instantaneous expected payoff rate at the verge of exiting the competition. $\tilde{\lambda}(t_2)H$ is the density of a success in stage 1; given the solution, the firm earns $p_1 + p_2$ if its opponent has not solved stage 1 and hence must have exited, and $\frac{p_1+p_2}{2}$ otherwise.

We can simplify the conditions as

$$t_1 = t_2 - \frac{\ln \frac{H(p_1+p_2)}{p_1 H - p_2 \mu}}{H+\mu} \text{ if } t_2 \ge \frac{\ln \frac{H(p_1+p_2)}{p_1 H - p_2 \mu}}{H+\mu}, \text{ and } = 0 \text{ otherwise}$$
$$e^{2Ht_2} = \frac{\alpha}{1-\alpha}\left(\frac{H(p_1+p_2)\left[\frac{\frac{1}{2}H-\mu}{H-\mu} + \frac{\frac{1}{2}H}{H-\mu}e^{(H-\mu)(t_2-t_1)}\right]}{c} + \frac{\mu - He^{(H-\mu)(t_2-t_1)}}{H-\mu}\right). \tag{3}$$

From (3), we can solve for a unique set of t_1, t_2[3]. $\frac{\ln \frac{H(p_1+p_2)}{p_1 H - p_2 \mu}}{H+\mu}$ is the value of ∇ defined in assumption A3.

We next prove that these necessary conditions are also sufficient. We begin by showing that each firm's instantaneous rate of expected payoff is a continuous function of t. Then, following the notion of sequential rationality in an equilibrium, we proceed by checking each firm's incentives backwards ($t \ge t_2,\ t \in [t_1, t_2),\ t < t_1$). This process consists of proving four consecutive claims: (1) exiting after t_2 is optimal; (2) a firm will never exit before t_2; (3) a firm prefers withholding the solution to disclosure between t_1 and t_2; and (4) a firm prefers disclosure to withholding the solution before t_1.

It is worth noting that, even though possible deviations from the equilibrium path are uncountable, they can be categorized into three types, which turn out to require only a simple discussion of off-equilibrium beliefs. The first type is to exit prematurely before t_2 or remain in the game even without solution to stage 1 after t_2. As we have assumed that exit from stage 1 is not observed by the opponent, this deviation will not alter the opponent's belief in equilibrium. The second type is to withhold the solution to stage 1 before t_1. Again, this is not observable by the opponent and will not affect its belief. The third type is to disclose the solution to stage 1 after t_1. This deviation is observable by the opponent, after which the two firms will simultaneously enter the competition for stage 2. Hence, the opponent's off-equilibrium belief always jumps to 1 at the instant of disclosure.

Theorem 1 highlights our distinction with respect to previous works, namely, that in general a technology spillover will not occur if the invention of the intermediate product occurs late. For most of the values of the model parameters, the length of the withhold region is positive.

Whenever a firm solves stage 1, it immediately realizes that stage 1 is solvable, but its decision of whether to disclose the solution depends on when this jump in beliefs takes place. If a firm succeeds early in the competition, there a long time remains before the opponent quits; provided that stage 1 is solvable, the opponent may simply also solve stage 1 before the exit point. Hence, the rational choice is to immediately disclose the solution such that p_1 is secured. Here, a technology spillover will occur, and the overall progress of R&D is indeed accelerated (compared to, for instance, the case in which the two firms work independently on the project without any technology spillover).

However, when no firm has solved stage 1 before the time threshold t_1, the dominant incentive changes. Now that little time remains before the exit point, withholding the solution becomes more appealing since the opponent is unlikely to succeed in stage 1 if it has not already. As a result, both firms would rather conceal their success and hope that the

[3] It can also be seen here that $t_1 > 0$ for a generic set of parameter values. For instance, fix some p_1 and p_2 that satisfy the model assumptions, and then enlarge both parameters by $n > 1$ times. When n is sufficiently large, $t_1 > 0$ in equilibrium. We also provide a numerical example in Section 4.4.

opponent will quit. A technology spillover will never happen in this time region, and a competition is no different from independently pursuing R&D.

Finally, we note here that Theorem 1, as well as subsequent main results, extend readily to the case with a positive but low cost for working on stage 2. A high stage-2 cost, however, may result in asymmetric equilibrium behavior even between homogeneous firms, as well as two exit thresholds – with and without solution to stage 1 – in time. When a firm has solved (and withheld the solution to) stage 1 but has not solved stage 2, it will form a belief about the opponent's probability of also having solved stage 1, which affects its own continuation payoff and its decision of whether to stay in the game (the latter is non-trivial when stage-2 cost is high). Thus when it is not worthwhile for both firms to work on stage 2 at the same time, there may exist multiple asymmetric equilibria and a symmetric equilibrium with mixed strategies.

4.2. Comparative statics

Based on the equilibrium conditions (3), we can determine how the model parameters affect both t_2, the length of time for which a firm remains in the game without solving stage 1, and $t_2 - t_1$, the length of time for which a firm withholds the solution to stage 1 prior to its opponent's exit point.

First, as H increases, $t_2 - t_1$ decreases. After a firm solves stage 1, its incentive to withhold the solution comes from the hope that the opponent will exit the game later. If the opponent is more likely to solve stage 1, the incentive is weakened, and thus, the length of the withhold region must in turn be smaller. However, t_2 may either increase or decrease in H at different values of H. The impact of a higher H on a firm's payoff is twofold: it increases a firm's expected payoff when stage 1 is solvable, but it also decreases a firm's belief that stage 1 is solvable. The first effect dominates the second when H increases from a low level, and the firm is willing to persist longer because it has more hope of solving stage 1. Nevertheless, the dominant effect switches when H is already high and continues increasing. Now, a firm will exit the game earlier because its belief drops rapidly.

Second, although higher p_1 and p_2 both increase t_2 due to the larger reward for the succeeding firm at either stage, they influence $t_2 - t_1$ in opposite ways. On the one hand, $t_2 - t_1$ decreases in p_1 because a firm facing a higher p_1 becomes more willing to secure p_1 by disclosure. On the other hand, $t_2 - t_1$ increases in p_2 because a firm facing a higher p_2 will wait longer to secure a higher likelihood of solving stage 2 before its opponent.

Finally, the effects of μ on t_2 and on $t_2 - t_1$ are both indeterminate. When a firm has not solved stage 1, a higher μ implies not only a higher expected payoff once a firm solves stage 1 before its opponent but also faster belief updating downwards; hence, the overall influence on the firm's willingness to persist in the game is jointly determined by μ and the other payoff-related parameters. When a firm has solved stage 1, a higher μ means that it can solve stage 2 faster and that its opponent solves stage 2 faster after stage 1. Thus, the firm's reaction depends on the value of p_2: with an increased μ, $t_2 - t_1$ increases when stage 2 is highly valuable and decreases when stage 2 brings a lesser reward.

Without discounting, there exists a notable discontinuity of equilibrium behavior at $\mu = 0$. When $\mu = 0$, the (unique) equilibrium is straight forward: each firm discloses the solution to stage 1 whenever it becomes available, and exits the competition at a certain time threshold if the firm has not solved stage 1. Since the payoff associated with the unsolvable stage 2 is zero, a withhold region with a positive length can never exist in equilibrium.

Now suppose that $\mu > 0$ and is small. The length of the withhold region is $\frac{\ln \frac{H(p_1+p_2)}{p_1 H - p_2 \mu}}{H+\mu}$, which is bounded away from 0. To see why this discontinuity takes place, consider a firm which has solved stage 1. However small μ is, as long as its opponent quits, the firm earns the entire $p_1 + p_2$ in the case of no discount. This payoff is significantly higher than its payoff from disclosing stage 1's solution, $p_1 + \frac{p_2}{2}$. When time approaches its opponent's exit threshold t_2, the firm faces a clear trade-off: the benefit of withholding the solution is the upward increment in expected payoff right after t_2, while the cost is the possibility that the opponent will solve stage 1 between now and t_2. The value of the former is bounded away from 0 (in particular, bounded below by $\frac{p_2}{2} e^{-H(t_2 - t_1)}$), and the value of the latter approaches 0 as time approaches t_2. Hence, the firm is always incentivized to withhold stage 1's solution for a positive length of time before t_2, as long as μ is positive.

With positive discounting, the above discontinuity vanishes. When $\mu = 0$, no firm chooses to withhold the solution; when $\mu > 0$, a positive-length withhold region emerges, but its length approaches 0 as $\mu \to 0$.

4.3. Robustness of equilibrium

Theorem 1 characterizes the unique equilibrium of the R&D competition game. There are four assumptions in the strategic environment that can be relaxed or altered to test the robustness of our result: the immediate technology spillover following disclosure that lead both firms to working on stage 2, the zero discount rate, the equal sharing rule when both firms have the solution to stage 1, and the number of firms. In this section, we will extend our equilibrium characterization to each case and show that with some variation, the disclose-withhold-exit pattern still persists.

4.3.1. Robustness to pure information spillover

Our previous analysis assumes that once a solution is disclosed, it immediately becomes publicly available; the opponent then automatically proceeds to the second stage. A corresponding practical scenario is that the opponent can work on the second stage using the intermediate product bought from the winner of the first stage without incurring a significant cost.

However, this is not always true: in some cases, purchasing a product only grants the buyer the right of use rather than the true technology, or the purchase itself may be considerably expensive due to patent protection or related legal issues.

This section studies the case in which a disclosure does not make the intermediate product public; rather, disclosure means that the succeeding firm simply announces a solution and claims the associated revenue. We call this the NFRA (No Free-Riding Assumption). The spillover here is purely informational, in the sense that the opponent knows that stage 1 is possible, but it does not enjoy any actual technological benefit. In other words, the opponent still has to solve stage 1 by itself before advancing to stage 2.

In our model, NFRA does not change anything except for the definition of disclosure. When only one firm solves stage 1 and discloses the solution, only the firm itself begins working on stage 2. The opponent, however, knows that stage 1 is possible, and $\tilde{\lambda}$ jumps to 1. Thus, the instantaneous payoff for the opponent researching on stage 1 is $Hdt\frac{p_2}{2} - cdt$. If $Hp_2 < 2c$, the opponent immediately exits, and hence, disclosure is the dominant strategy. If $Hp_2 \geq 2c$, the opponent remains in the competition.

Now, we characterize the equilibrium under NFRA.

Proposition 1. *Under NFRA: If $Hp_2 \geq 2c$, the unique equilibrium still exhibits a symmetric disclose-withhold-exit pattern; otherwise, the unique equilibrium exhibits a symmetric disclose-exit pattern.*

In the former case, the values of cutoffs in time are different from those in Theorem 1. In particular, the withhold region is shorter.

Proof. See Appendix C. □

The basic intuition for this result is the same as that for Theorem 1. However, as disclosure does not automatically make the intermediate product available but simply increases the opponent's belief $\tilde{\lambda}$, the cost of disclosure is lower under NFRA. Therefore, the withhold region is shortened.

4.3.2. Robustness to Discount

We assume now that the discount rate $r > 0$ and focus on symmetric equilibria, i.e. the firms take identical actions at the same time given the same state (whether or not they have the solution to stage 1). With a positive discount rate, Lemmas 1 and 2 still hold with proofs unchanged. However, we need to modify the statement and the proof of Lemma 3.

Lemma 3'. *In every symmetric equilibrium, the behavior of each firm exhibits a disclose-withhold-exit pattern: there exist cutoffs t_1, t_2 such that at time t, the firm*

$$\begin{cases} \textit{discloses the solution to stage 1 if } t \in [0, t_1) \\ \textit{withholds the solution to stage 1 if } t \in [t_1, t_2) \\ \text{exits if } t \in [t_2, \infty) \textit{ when it has not solved stage 1.} \end{cases}$$

Proof. See Appendix D. □

The R&D competition game with positive discount rate has a unique symmetric equilibrium, whose characterization is given by Theorem 1 ′ below.

Theorem 1'. *The R&D competition game has a unique symmetric equilibrium. The equilibrium is a disclose-withhold-exit strategy profile characterized by cutoffs $t_1, t_2 \geq 0$ such that $t_1^i = t_1^j = t_1$ and $t_2^i = t_2^j = t_2$ and that at time t, each firm*

$$\begin{cases} \textit{discloses the solution to stage 1 if } t \in [0, t_1) \\ \textit{withholds the solution to stage 1 if } t \in [t_1, t_2) \\ \textit{exits if } t \in [t_2, \infty) \textit{ when it has not solved stage 1.} \end{cases}$$

Proof. See Appendix E. □

Furthermore, we can show that there never exists an asymmetric equilibrium where $t_1^i < t_1^j < t_2^j < t_2^i$, i.e. the firm who starts withholding the solution earlier will exit the competition later than its opponent. When the discount rate r is sufficiently small, the other possible type of asymmetric equilibrium, i.e. one in which $t_1^i < t_1^j < t_2^i < t_2^j$, cannot exist either.

4.3.3. Robustness to policy rules

In practice, two policy rules are prevalent for the division of related profits once more than one competing firms have invented the same product or technology: the first-to-invent rule and the first-to-file rule. Our previous model assumption of equally sharing p_1 can be regarded as a case with friction under the first-to-file rule: due to uncertainty and legal complexities, disclosure yields $\frac{1}{2}p_1$ when both firms already have the solution to stage 1.

Under the first-to-invent rule, equilibrium characterization is similar to Theorem 1, with the variation that the disclose region always has length zero when $r = 0$, and only has a positive length when r is large. Indeed, when there is no or a very low discount, disclosure brings virtually no difference to a firm's profit from stage 1, and hence withholding is the dominant action.

We will then mainly discuss the frictionless case under the first-to-file rule: whichever firm to first disclose the solution to stage 1 obtains the entire p_1. This corresponds to the case where there is absolutely no uncertainty or friction in dividing the intermediate reward. We focus on symmetric equilibria.

Note that Lemmas 1 and 2 are still straightforward in this case. In addition, Lemma 3 is trivially true: if that there were multiple withhold regions in some equilibrium, a firm would always prefer disclosure an instant before the end of the first withhold region, a contradiction.

The following result shows that the first-to-file rule results in simple equilibrium behavior: there cannot be any disclosure.

Proposition 2. *Under the first-to-file rule, every symmetric equilibrium exhibits a withhold-exit pattern.*

Proof. See Appendix F. □

Proposition 2 gives a somehow surprising statement since the first-to-file rule is supposed to encourage disclosure. In fact, the first-to-file rule implies nonexistence of symmetric equilibria under certain conditions, for instance relatively large r under assumption A2: there are incentives to disclose the solution early in the game and withhold the solution when time approaches the exit point, but to make withholding the solution preferable at the beginning of the withhold region requires a condition that violates assumption A2[4]. There may be asymmetric equilibria in such cases, which lie outside the scope of discussion by this paper[5]. Under other conditions, for instance relatively large p_1 and small r under assumption A2, or relatively large c, there exists a unique symmetric withhold-exit equilibrium.

When the division of p_1 with both firms having the solution is not perfectly first-to-file due to possible friction, as we have argued, equilibrium characterization is in between the above extreme case and our original model. If the first firm to disclose the solution gets βp_1, $\beta \in [\frac{1}{2}, 1]$, the range of parameters for the existence of a (unique) symmetric equilibrium and that of disclosure in the symmetric equilibrium expands as β decreases from 1 towards $\frac{1}{2}$. At $\beta = \frac{1}{2}$, as our previous results have shown, a unique symmetric equilibrium always exists while the length of the disclose region depends on the research cost c.

4.3.4. Robustness tonumber of firms

Our analysis can be readily extended to the case with n ex-ante identical firms. Assume the following version of assumption A3 with multiple firms (with "m" standing for multiple firms):

A3(m) Let $A = e^{-H(t_2-t_1)}$ and $B = \int_0^{t_2-t_1} He^{-Hs-\mu(t_2-t_1-s)}ds = \frac{H}{H-\mu}(e^{-\mu(t_2-t_1)} - e^{-H(t_2-t_1)})$. The condition below is satisfied:

$$\frac{A^2((A+B)^{n-1} - A^{n-1}))}{Be^{nHt_1}} \le \frac{1-\alpha}{\alpha}.$$

The following result shows the existence of a unique symmetric equilibrium.

Proposition 3. *The R&D competition game has a unique symmetric equilibrium, which exhibits a disclose-withhold-exit pattern.*

Proof. See Appendix G. □

The challenges in proving this result are mainly technical, which we choose not to elaborate here. In the proof, we discuss how to resolve several additional issues that arise when the number of firms becomes arbitrary.

4.4. Welfare analysis

In this section, we analyze the effect of uncertainty on social welfare. In particular, we are interested in whether competition always facilitates efficiency: under uncertainty, is a two-firm competition always socially preferred to a one-firm monopoly? To provide a tractable answer, we assume that the society faces the same costs as the firms in developing the two stages but values a success in each stage at $p_1^S > p_1$ and $p_2^S > p_2$ respectively. Thus, if there is only one firm, the society always prefers the firm to persist in researching stage 1 longer than it would in equilibrium. We also assume that $p^S = p_1^S + p_2^S$ is sufficiently large so that in a two-firm competition the socially optimal total research time devoted to stage 1 is larger than $2t_2$. In practical scenarios, this setting refers to research projects with large positive externalities, such as national defense or drug development.

When there is only one firm, information is perfect: the firm always knows whether it has solved stage 1. With two competing firms, however, information is imperfect in the sense that no firm knows whether its opponent has solved stage 1 once they enter the withhold region. In general, the imperfect information has three effects that work in different directions regarding social welfare.

Effect 1: A firm's expected payoff from solving stage 1 decreases. For a single firm, a success in stage 1 delivers the entire reward; however, for two competing firms in the withhold region, a firm fears that the competitor has already solved the

[4] If we start with the opposite assumption to A2, namely $p_1 + \frac{\mu}{2\mu+r}p_2 \ge \frac{\mu}{\mu+r}(p_1+p_2)$, there exists a unique symmetric disclose-exit equilibrium.

[5] Section 5 provides a discussion on asymmetric equilibrium behavior of firms that are heterogeneous *per se*, in cost or in research ability.

first stage, meaning that a success in stage 1 will only yield $\frac{p_1+p_2}{2}$. From this perspective, a firm is less willing to remain in the game and this effect is detrimental to social welfare.

Effect 2: There is possible waste of effort. During the withhold region, a firm does not know whether the opponent has finished stage 1. If the opponent has actually done so, then the firm's further research on stage 1 is wasted from a social welfare perspective. Hence, this effect is detrimental to social welfare.

Effect 3: A firm's belief about stage 1 being solvable decreases more slowly. For a single firm, all the unfruitful effort reduces its belief $\tilde{\lambda}(t)$. However, for two competing firms in the withhold region, a firm is uncertain about the result of the competitor's effort, making it less pessimistic about the difficulty of stage 1. From this perspective, a firm is more willing to remain in the game and this effect is beneficial to social welfare.

When the first two effects combined are larger than (equal to, smaller than) the third, competition hurts (does not change, improves) social welfare. Our first result below states a clear characterization of the socially optimal choice between monopoly and competition when there is no discounting in payoff.

Theorem 2. *Assume that $r = 0$. From a welfare perspective, there exists a single cutoff c^* such that*

$$\begin{cases} the\ society\ prefers\ a\ one-firm\ monopoly\ if\ c > c^* \\ the\ society\ is\ indifferent\ if\ c = c^* \\ the\ society\ prefers\ a\ two-firm\ competition\ c < c^*. \end{cases}$$

Furthermore, if $p^S = \infty$, $c^* = \frac{H(p_1+p_2)}{2}$.

Proof. See Appendix H. □

The intuition behind this result is as follows. When c is high, a firm in a competition exits sooner, i.e., its belief $\tilde{\lambda}(t)$ remains relatively high when exiting. Thus, by the end of the withhold region, as time passes, the firm is not yet particularly concerned about the possibility that the task is impossible (Effect 3); rather, it is scared away because its opponent is increasingly likely to have finished the first stage (Effect 1). As Effect 1 plays the decisive role, two firms in a competition are less persistent than one in a monopoly. As the society prefers higher persistence than in equilibrium, plus competition may create waste of costly effort due to Effect 2, a monopoly generates higher social welfare in this case.

However, when c is low, a firm exits later when its belief $\tilde{\lambda}(t)$ is relatively low. Thus, by the end of the withhold region, the firm is less concerned that its opponent may have solved stage 1 but is more concerned that stage 1 may not be solvable after all. In contrast to the above situation, Effect 1 is now relatively small, and Effect 3 dominates. As we have discussed, $\tilde{\lambda}(t)$ evolves more slowly due to imperfect information, and hence, two firms in a competition are more persistent than one in a monopoly.

With the discount rate $r > 0$, the welfare comparison result is an ϵ-approximation of Theorem 2, as stated below.

Corollary 1. *For any $\epsilon > 0$, there exists $\bar{r} > 0$ such that for all $r < \bar{r}$,*

$$\begin{cases} the\ society\ prefers\ a\ one-firmmonopoly\ if\ c > c^* + \epsilon \\ the\ society\ prefers\ a\ two-firm\ competition\ if\ c < c^* - \epsilon. \end{cases}$$

At the end of this section, we provide a simple numerical example to illustrate the key incentives in the R&D competition game as well as the welfare comparison between competition and monopoly.

Numerical Example. We assume that $H = 1$ and $\mu = 0.5$, prior $\alpha = 0.8$, cost of research $c = 0.8$, discount rate $r = 0$, private revenue $p_1 = 1$ and $p_2 = 0.2$, and social benefit $p^S = \infty$. The numerical result is depicted by Figs. 2–5 (see Appendix J).

Fig. 2 shows the trajectory of the firm's belief regarding the difficulty of the first stage, i.e., $\tilde{\lambda}(t)$; Fig. 3 illustrates the comparison of a firm's revenue from "disclosing the solution now" and from "withholding the solution until the end", provided that the firm has invented the intermediate product. The firm exhibits disclose-withhold-exit behavior with cutoffs $t_1 = 0.103$ and $t_2 = 0.294$, i.e., the disclose region is $[0, 0.103)$, the withhold region is $[0.103, 0.294)$, and the exit point is 0.294. Thus, the maximum time spent on research by the two firms without solving stage 1 is 0.588. For a monopolist, the corresponding time is 0.693. From the society's perspective, a monopoly is preferred.

Fig. 4 shows the probability density functions (pdfs) of the whole project being successful. The kinks in the curves mark important time thresholds. The one for monopoly occurs at the firm's exiting point, where the pdf suddenly turns downward because the firm may simply abandon its R&D effort afterwards. For competition, the second kink which corresponds to t_2 occurs for a similar reason. The first kink corresponds to t_1, after which the pdf increases significantly slower because both firms start withholding their solution to stage 1 if any, and the progress of the whole project gets hindered compared to the disclose region. In Fig. 5 we rescale the curve for competition so that its time axis is stretched to twice as the one for monopoly. It is then clear that the two curves coincide up to time $2t_1$, after which the pdf under competition increases slower due to the existence of the withhold region.

For comparison purposes, if we lower c to 0.2, i.e., smaller than $\frac{H(p_1+p_2)}{2}$, the cutoffs become $t_1 = 1.346$ and $t_2 = 1.537$; in the two-firm competition, the maximum time spent on research without solving stage 1 is 3.074. However, in such a case, a monopolist will exit at $t = 2.996$. From the society's perspective, a competition is preferred.

5. Asymmetric competition

Our basic model applies to rivalries in which the opponents are well matched in management efficiency and research ability. However, in many other competitions, the opponents are not equal. An overdog either manages its costs more efficiently or simply assemblies a more prolific R&D team. Either form of superiority affects the incentives of the firm and its opponent and leads to a potentially different equilibrium.

We study three model variations with asymmetric competitors. We use $i, j \in \{A, B\}$, to denote the two firms. Without loss of generality, we assume that firm A is stronger than firm B, and we will call A the "Alpha" and B the "Beta." To keep notations simple, we assume that $r = 0$.

5.1. Asymmetric Costs

In this section, we analyze the case of asymmetric costs: firm i pays a cost per unit of time $c^i > 0$ ($c^j > 0$ for firm j) during stage 1, and $c^A < c^B$. The game otherwise remains the same as in Section 3. We impose a different version of assumptions A1 and A3 here (with "a" standing for asymmetric firms):

A1(a). $\alpha H(p_1 + p_2) > c^A$.

A3(a). The following inequality is satisfied:

$$c^B \le \frac{H\frac{p_1+p_2}{2}\left(\frac{\mu}{H+\mu}e^{-H\nabla} + \frac{H}{H-\mu}e^{-\mu\nabla} - \frac{\mu}{H-\mu}e^{-H\nabla}\right)}{e^{H\nabla} + \frac{H}{H-\mu}e^{-\mu\nabla} - \frac{\mu}{H-\mu}e^{-H\nabla}},$$

where $\nabla = \frac{\ln\frac{H(p_1+p_2)}{p_1 H - p_2\mu}}{H+\mu} > 0$.

The trajectory of i's private belief, denoted $\tilde{\lambda}^i(t)$, is unchanged if the opponent is in a disclose region or a withhold region. If time t is past the opponent's exit point t_n^j following a withhold region $[t_{n-1}^j, t_n^j)$, the trajectory becomes

$$\tilde{\lambda}^i(t) = \frac{\alpha e^{-2Ht_{n-1}^j - H(t-t_n^j)}\left(e^{-H(t_n^j - t_{n-1}^j)} + e^{-\mu(t-t_n^j)}\int_0^{t_n^j - t_{n-1}^j} He^{-sH}e^{-(t_n^j - t_{n-1}^j - s)\mu}ds\right)}{\alpha e^{-2Ht_{n-1}^j - H(t-t_n^j)}\left(e^{-H(t_n^j - t_{n-1}^j)} + e^{-\mu(t-t_n^j)}\int_0^{t_n^j - t_{n-1}^j} He^{-sH}e^{-(t_n^j - t_{n-1}^j - s)\mu}ds\right) + 1 - \alpha}.$$

To begin our analysis, it is straightforward that Lemmas 1–3 remain valid. We characterize the unique equilibrium under asymmetric costs below.

Theorem 3. *There exists a unique equilibrium in the R&D competition game with asymmetric costs. Let $[t_1^A, t_2^A)$ be the Alpha's withhold region and $[t_1^B, t_2^B)$ be the Beta's withhold region in equilibrium, and we have $t_1^A < t_1^B \le t_2^B < t_2^A$.*

Proof. See Appendix I. □

The equilibrium property that one firm's withhold region must be either a subset or a superset of the other's withhold region stems from each firm's incentive at the beginning of its withhold region. Provided that a firm has solved stage 1, its withhold region begins at the instant when it is indifferent between disclosing and withholding the solution. For the firm that enters a withhold region first, it faces exactly the same trade-off as a firm in the previous basic model. Therefore, the time length between the start of its withhold region and the opponent's exit point must be equal to $\nabla = \frac{\ln\frac{H(p_1+p_2)}{p_1 H - p_2\mu}}{H+\mu}$, the same as $t_2 - t_1$ before. For the other firm, its expected payoffs from both disclosure and withholding the solution decrease because there is a possibility that the opponent has solved stage 1 by this point. However, if this is the case, the firm's payoff will be $\frac{p_1+p_2}{2}$ whether or not it chooses disclosure; hence, when the firm decides when to start its withhold region, it is essentially comparing its payoff from the two options provided that the opponent has not solved stage 1, which again reduces to the same problem as in the basic model[6]. Now that the difference between one firm's exit point and the starting point of the other's withhold region must be identical across firms, one withhold region must lie within the other. Fig. 6 (see Appendix J) depicts a typical equilibrium: on the one hand, the Alpha begins to withhold the solution earlier, at t_1^A, than the Beta, at t_1^B; on the other hand, the Alpha exits the game later, at t_2^A, than the Beta, at t_2^B.

Theorem 3 indicates that an exogenous cost advantage will always result in an endogenous information advantage. Whichever firm exits later will observe for an additional period of time whether its opponent has solved stage 1 and hence has a lower belief than its opponent in the opponent's withhold region. In contrast, the expected payoff for this firm upon solving stage 1 or 2 is higher than that for its opponent in the opponent's withhold region, as the firm is more convinced that the opponent has not succeeded in stage 1 based on the additional time of observation. In the proof, we find that the first effect will ultimately dominate the second as time approaches the opponent's exit point. In other words, the firm that exits later must be the firm with a lower cost, to maintain its incentive of persisting in the game despite the lower belief. Thus the cost-efficient firm observes more, exits later, and is more likely to win the competition.

[6] We provide a detailed derivation of this time length in the proof of Theorem 3 in Appendix I. This property of equal time lengths holds for generic cases in which $0 < t_1^A < t_1^B < t_2^B < t_2^A$ but may be violated in extreme scenarios in which t_1^A or t_1^B is zero or $t_1^B = t_2^B$.

5.2. Asymmetric research abilities

In this section, we analyze firms that differ in research ability. Instead of sharing the same success arrival rate, now firm i's Poisson rate of success, while still independent of firm j's, is equal to $H^i\lambda$ (resp. $H^j\lambda$ for firm j). We assume that $\lambda \in \{1, 0\}$ and $H^A > H^B$. In other words, although the two firms face the same probability that stage 1 is unsolvable, the stronger firm has a higher arrival rate of success when stage 1 is solvable. The cost c is assumed to be the same for both firms.

The trajectory of $\tilde{\lambda}^i(t)$ is as follows:

If the opponent is in a disclose region,

$$\tilde{\lambda}^i(t) = \frac{\alpha e^{-(H^i+H^j)t}}{\alpha e^{-(H^i+H^j)t} + 1 - \alpha}.$$

If the opponent is in a withhold region starting from t^j_k, i.e., $t \in [t^j_k, t^j_{k+1}]$, the trajectory becomes

$$\tilde{\lambda}^i(t) = \frac{\alpha e^{-H^j t^j_k - H^i t}[e^{-H^j(t-t^j_k)} + \int_0^{t-t^j_k} H^j e^{-sH^j} e^{-(t-t^j_k-s)\mu} ds]}{\alpha e^{-H^j t^j_k - H^i t}(e^{-H^j(t-t^j_k)} + \int_0^{t-t^j_k} H^j e^{-sH^j} e^{-(t-t^j_k-s)\mu} ds) + 1 - \alpha}.$$

If the opponent's exit point is $t^j_n < t$ and it withholds the solution in $[t^j_{n-1}, t^j_n]$, the trajectory becomes

$$\tilde{\lambda}^i(t) = \frac{\alpha e^{-I_j t^j_{n-1} - I_i t}[e^{-I_j(t^j_n - t^j_{n-1})} + \int_0^{t^j_n - t^j_{n-1}} I_j e^{-I_j s} e^{-(t-t^j_{n-1}-s)\mu} ds]}{\alpha e^{-I_j t^j_{n-1} - I_i t}[e^{-I_j(t^j_n - t^j_{n-1})} + \int_0^{t^j_n - t^j_{n-1}} I_j e^{-I_j s} e^{-(t-t^j_{n-1}-s)\mu} ds] + 1 - \alpha}.$$

Lemmas 1–3 still hold in this scenario. As before, we let t^i_1, t^j_1 be the starting points of the two firms' withhold regions and t^i_2, t^j_2 be their exit points. Assuming that $t^i_2 \geq t^j_2$, we have a set of necessary and sufficient conditions for an equilibrium:

$$\begin{aligned}
t^i_1 &= t^j_2 - \frac{\ln \frac{H^j(p_1+p_2)}{p_1 H^j - p_2\mu}}{H^j + \mu} \text{ if } t^j_2 \geq \frac{\ln \frac{H^j(p_1+p_2)}{p_1 H^j - p_2\mu}}{H^j + \mu}; \quad = 0 \text{ otherwise} \\
t^j_1 &= t^i_2 - \frac{\ln \frac{H^i(p_1+p_2)}{p_1 H^i - p_2\mu}}{H^i + \mu} \text{ if } t^i_2 \geq \frac{\ln \frac{H^i(p_1+p_2)}{p_1 H^i - p_2\mu}}{H^i + \mu}; \quad = 0 \text{ otherwise} \\
c &= H^i \frac{(p_1+p_2)e^{-H^j t^j_2 - H^i t^i_2} + \frac{p_1+p_2}{2} e^{-H^j t^j_1 - H^i t^i_2} \int_0^{t^j_2 - t^j_1} H^j e^{-H^j s} e^{-(t^i_2 - t^j_1 - s)\mu} ds}{e^{-H^j t^j_2 - H^i t^i_2} + e^{-H^j t^j_1 - H^i t^i_2} \int_0^{t^j_2 - t^j_1} H^j e^{-H^j s} e^{-(t^i_2 - t^j_1 - s)\mu} ds + \frac{1-\alpha}{\alpha}} \\
c &= H^j \frac{\begin{array}{c}(p_1+p_2)e^{-H^i t^j_2 - H^j t^j_2}(1 - \frac{1}{2}\int_0^{t^i_2 - t^j_2} H^i e^{-H^i s} e^{-\mu s} ds)_c \\ + \frac{p_1+p_2}{2} e^{-H^i t^i_1 - H^j t^j_2} \int_0^{t^j_2 - t^i_1} H^i e^{-H^i s} e^{-(t^j_2 - t^i_1 - s)\mu} ds\end{array}}{e^{-H^i t^j_2 - H^j t^j_2} + e^{-H^i t^i_1 - H^j t^j_2} \int_0^{t^j_2 - t^i_1} H^i e^{-H^i s} e^{-(t^j_2 - t^i_1 - s)\mu} ds + \frac{1-\alpha}{\alpha}}.
\end{aligned} \tag{4}$$

From (4), we know that the difference between the starting point of the Alpha's withhold region and the Beta's exit point is longer than its counterpart. Intuitively, the faster the opponent is able to solve stage 1, the less time a firm is willing to wait. However, now there is no definite answer to whether the Alpha or the Beta will exit first. For instance, fix all the parameters except H^i, and assume that $\alpha H^j \frac{p_1+p_2}{2} > c$. Since $\frac{p_1+p_2}{2}$ is the minimum payoff that firm j obtains if it solves stage 1 before firm i solves stage 2, we can then derive a uniform positive lower bound of t^j_2 in every equilibrium. Let $\underline{t}^j > 0$ denote this lower bound, and consider different values of H^i. If $H^i < H^j$ and H^i is sufficiently small, e.g., when $\alpha H^i(p_1+p_2) \approx c$, $t^i_2 \approx 0 < \underline{t}^j$ and thus the Alpha exits later than the Beta in every equilibrium. However, if $H^i > H^j$ and H^i is sufficiently large, e.g., when $\frac{\alpha e^{-H^i \frac{\underline{t}^j}{2}}}{\alpha e^{-H^i \frac{\underline{t}^j}{2}} + 1 - \alpha} H^i(p_1+p_2) < c$, $t^i_2 < \frac{\underline{t}^j}{2} < \underline{t}^j$, and thus, the Alpha exits earlier than the Beta in every equilibrium. Compared with the previous section on asymmetric costs, our analysis shows that although lower cost and higher research capacity both serve as indicators of a stronger firm in competition, they have different implications for the length of a firm's withhold region and its exit time.

5.3. Asymmetric commercial abilities

Asymmetry between firms may also arise from commercial abilities. Suppose that the two firms have different Poisson rates of success for stage 2, μ^i and μ^j, and that $\mu^A > \mu^B$. Without loss of generality, assume that firm i enters the withhold region first at t^i_1 and exits at t^i_2, while firm j enters the withhold region at $t^j_1 \geq t^i_1$ and exits at t^j_2. Applying the same method in Part I of Theorem 1's proof (Appendix B), we have

$$
\begin{aligned}
& e^{-H(t_2^j-t_1^i)}e^{-\mu_1(t_2^j-t_1^i)}(p_1+p_2)+\int_{t_1^i}^{t_2^j}\mu_1 e^{-\mu_1(s-t_1^i)}e^{-H(s-t_1^i)}ds(p_1+p_2) \\
& +\int_{t_1^i}^{t_2^j} He^{-H(s-t_1^i)}e^{-\mu_1(s-t_1^i)}ds\left(\frac{p_1}{2}+\frac{\mu_1 p_2}{\mu_1+\mu_2}\right)=p_1+\frac{\mu_1 p_2}{\mu_1+\mu_2} \\
\Rightarrow & \left(\frac{\mu_1}{H+\mu_1}+\frac{H}{H+\mu_1}e^{-(H+\mu_1)(t_2^j-t_1^i)}\right)(p_1+p_2) \\
& +\frac{H}{H+\mu_1}(1-e^{-(H+\mu_1)(t_2^j-t_1^i)})\left(\frac{p_1}{2}+\frac{\mu_1 p_2}{\mu_1+\mu_2}\right)=p_1+\frac{\mu_1 p_2}{\mu_1+\mu_2} \\
\Rightarrow & t_2^j-t_1^i=\frac{\ln\frac{H(p_1+\frac{2\mu_2}{\mu_1+\mu_2}p_2)}{Hp_1-\frac{2\mu_1\mu_2}{\mu_1+\mu_2}p_2}}{H+\mu_1},
\end{aligned}
$$

and

$$
\begin{aligned}
& \frac{\begin{array}{c}\left(\frac{p_1}{2}+\frac{\mu_2 p_2}{\mu_1+\mu_2}\right)\int_{t_1^i}^{t_1^j}He^{-H(s-t_1^i)}e^{-\mu_1(t_1^j-s)}ds+e^{-H(t_1^j-t_1^i)}[(p_1+p_2)e^{-H(t_2^i-t_1^j)}e^{-\mu_2(t_2^i-t_1^j)} \\ +(p_1+p_2)\int_{t_1^j}^{t_2^i}\mu_2 e^{-H(s-t_1^j)}e^{-\mu_2(s-t_1^j)}ds+\left(\frac{p_1}{2}+\frac{\mu_2 p_2}{\mu_1+\mu_2}\right)\int_{t_1^j}^{t_2^i}He^{-H(s-t_1^j)}e^{-\mu_2(s-t_1^j)}ds]\end{array}}{e^{-H(t_1^j-t_1^i)}+\int_{t_1^i}^{t_1^j}He^{-H(s-t_1^i)}e^{-\mu(t_1^j-s)}ds} \\
& \qquad =\frac{(p_1+\frac{\mu_2 p_2}{\mu_1+\mu_2})e^{-H(t_1^j-t_1^i)}+\left(\frac{p_1}{2}+\frac{\mu_2 p_2}{\mu_1+\mu_2}\right)\int_{t_1^i}^{t_1^j}He^{-H(s-t_1^i)}e^{-\mu_1(t_1^j-s)}ds}{e^{-H(t_1^j-t_1^i)}+\int_{t_1^i}^{t_1^j}He^{-H(s-t_1^i)}e^{-\mu_1(t_1^j-s)}ds} \\
\Rightarrow & \begin{array}{c} e^{-H(t_1^2-t_1^i)}[(p_1+p_2)e^{-H(t_2^i-t_1^j)}e^{-\mu_2(t_2^i-t_1^j)}+(p_1+p_2)\int_{t_1^j}^{t_2^i}\mu_2 e^{-H(s-t_1^j)}e^{-\mu_2(s-t_1^j)}ds \\ +\left(\frac{p_1}{2}+\frac{\mu_2 p_2}{\mu_1+\mu_2}\right)\int_{t_1^j}^{t_2^i}He^{-H(s-t_1^j)}e^{-\mu_2(s-t_1^j)}ds]=(p_1+\frac{\mu_2 p_2}{\mu_1+\mu_2})e^{-H(t_1^j-t_1^i)}\end{array} \\
\Rightarrow & \begin{array}{c} (p_1+p_2)e^{-H(t_2^i-t_1^j)}e^{-\mu_2(t_2^i-t_1^j)}+(p_1+p_2)\int_{t_1^j}^{t_2^i}\mu_2 e^{-H(s-t_1^j)}e^{-\mu_2(s-t_1^j)}ds \\ +\left(\frac{p_1}{2}+\frac{\mu_2 p_2}{\mu_1+\mu_2}\right)\int_{t_1^j}^{t_2^i}He^{-H(s-t_1^j)}e^{-\mu_2(s-t_1^j)}ds]=p_1+\frac{\mu_2 p_2}{\mu_1+\mu_2}\end{array} \\
\Rightarrow & t_2^i-t_1^j=\frac{\ln\frac{H(p_1+\frac{2\mu_1}{\mu_1+\mu_2}p_2)}{Hp_1-\frac{2\mu_1\mu_2}{\mu_1+\mu_2}p_2}}{H+\mu_2}.
\end{aligned}
$$

To characterize equilibrium behavior under asymmetric abilities in the commercial stage, consider first an extreme case: stage 2 is considerably easy for firm i, i.e. $\mu_1 \gg 0$; at the same time, it is extremely difficult for firm j, i.e. $\mu_2 \approx 0$. It is clear that (1) firm i's exit point t_2^i is significantly later than firm j's exit point t_2^j, and (2) firm j will never find it optimal to withhold the solution to stage 1. Therefore, we can solve for a unique combination of $t_1^j = t_2^j$, t_1^i and t_2^i as follows:

$$
\begin{aligned}
\tilde{\lambda}(t_1^j)H\frac{(p_1+\frac{\mu_2 p_2}{\mu_1+\mu_2})e^{-H(t_1^j-t_1^i)}+\left(\frac{p_1}{2}+\frac{\mu_2 p_2}{\mu_1+\mu_2}\right)\int_{t_1^i}^{t_1^j}He^{-H(s-t_1^i)}e^{-\mu(t_1^j-s)}ds}{e^{-H(t_1^j-t_1^i)}+\int_{t_1^i}^{t_1^j}He^{-H(s-t_1^i)}e^{-\mu(t_1^j-s)}ds} &= c \\
t_1^j-t_1^i &= \frac{\ln\frac{H(p_1+\frac{2\mu_2}{\mu_1+\mu_2}p_2)}{Hp_1-\frac{2\mu_1\mu_2}{\mu_1+\mu_2}p_2}}{H+\mu_1} \\
\tilde{\lambda}(t_2^i)H(p_1+p_2) &= c.
\end{aligned}
$$

Now let the difference between μ_1 and μ_2 shrink, which first causes $t_2^1 - t_2^2$, the length of time between the two firms' exit points, to decrease. When $t_2^1 - t_2^2$ becomes sufficiently small, firm 2 will find it beneficial to start withholding the solution to stage 1 shortly before firm 1's exit point. Hence, the equilibrium resembles that under asymmetric research costs, with firm 2 analogous to a high-cost firm and firm 1 a low-cost firm.

6. Conclusion

Uncertainty in the difficulty of research is a common and important feature of an R&D process. In this paper, we have proposed a model to analyze how uncertainty affects a firm's strategic behavior in a two-stage R&D competition. The competing firms are free to choose whether and when to disclose their solution to the intermediate stage and whether and when to exit the entire competition. We find that this game has a unique symmetric equilibrium featuring two cutoffs of time: each firm will disclose its solution if it solves the intermediate stage by the first cutoff, withhold its solution between the two cutoffs, and exit if it has not solved the intermediate stage by the second cutoff. When the R&D project is sufficiently valuable to the society, we show that a competition is not always the desired scheme: the society may benefit from assigning the project to a single firm. When the firms are heterogeneous, asymmetric costs always result in an information advantage for the cost efficient firm, while asymmetric research abilities may entail opposite predictions of equilibrium behavior given different parameters.

We believe that this paper may open the way for richer studies on R&D competitions with uncertainty and related policy issues. One natural extension of the model is to study firms' learning dynamics with heterogeneous priors. Furthermore, in some practical scenarios, there may be more than one research path to reach the final stage, and it would be interesting to explore how competing firms choose to experiment and learn about distinct paths.

Appendix A. Proof of Lemma 3

Clearly, full disclosure is never an equilibrium, and thus, in every equilibrium, there must be at least one withhold region.

Firm i's problem. Assume that firm i is the first firm to enter a withhold region at t_1^i and that it plans to return to a disclose region at t_2^i. At time t_1^i, as its opponent weakly prefers disclosure by assumption, firm i's payoff from disclosure is $p_1 + \frac{1}{2}p_2$. Since a firm will withhold the solution after its opponent's exit point, firm i will not shift to disclosure at t_2^i if firm j exits before t_2^i. Thus, firm j's exit time is weakly larger than t_2^i, and firm i's payoff from withholding the solution is

$$
\begin{aligned}
& e^{-H(t_2^i-t_1^i)}e^{-\mu(t_2^i-t_1^i)}(p_1+\frac{p_2}{2}) + \int_{t_1^i}^{t_2^i} \mu e^{-\mu(s-t_1^i)}e^{-H(s-t_1^i)}ds(p_1+p_2) \\
& + \int_{t_1^i}^{t_2^i} He^{-H(s-t_1^i)}e^{-\mu(s-t_1^i)}ds\frac{(p_1+p_2)}{2} \\
= & \left[-\frac{\mu}{H+\mu}e^{-H(s-t_1^i)-\mu(s-t_1^i)}\Big|_{t_1^i}^{t_2^i}\right](p_1+p_2) + e^{-H(t_2^i-t_1^i)-\mu(t_2^i-t_1^i)}(p_1+\frac{p_2}{2}) \\
& + \left[-\frac{H}{H+\mu}e^{-H(s-t_1^i)-\mu(s-t_1^i)}\Big|_{t_1^i}^{t_2^i}\right]\frac{(p_1+p_2)}{2} \\
= & \ p_1\left[\frac{\frac{1}{2}H}{H+\mu}e^{-(H+\mu)(t_2^i-t_1^i)} + \frac{\frac{1}{2}H+\mu}{H+\mu}\right] + p_2\left[\frac{-\frac{1}{2}\mu}{H+\mu}e^{-(H+\mu)(t_2^i-t_1^i)} + \frac{\frac{1}{2}H+\mu}{H+\mu}\right].
\end{aligned}
$$

At cutoff t_1^i, firm i should be indifferent between disclosing and withholding the solution:

$$
\begin{aligned}
p_1\frac{\frac{1}{2}H}{H+\mu}[e^{-(H+\mu)(t_2^i-t_1^i)}-1] + p_2\frac{\frac{1}{2}\mu}{H+\mu}[1-e^{-(H+\mu)(t_2^i-t_1^i)}] = 0 \\
p_1\frac{\frac{1}{2}H}{H+\mu} = p_2\frac{\frac{1}{2}\mu}{H+\mu},
\end{aligned}
$$

which contradicts our assumption $p_1H > p_2\mu$. Therefore, firm i, i.e., the firm that first enters a withhold region, must exit at the end of the withhold region.

Firm j's problem. Assume that firm i is the first firm to enter a withhold region at t_1^i and that it plans to exit at t_2^i. Suppose that firm j plans to begin withholding the solution at t_1^j and to resume to disclosure at time t_2^j. By the same argument as in firm i's problem, $t_1^j, t_2^j \in [t_1^i, t_2^i]$. Then, at $t \in [t_1^j, t_2^j]$, firm j's payoff from disclosure is

$$
\frac{(p_1+\frac{p_2}{2})e^{-H(t-t_1^i)} + \frac{(p_1+p_2)}{2}\int_{t_1^i}^t He^{-H(s-t_1^i)}e^{-\mu(t-s)}ds}{e^{-H(t-t_1^i)} + \int_{t_1^i}^t He^{-H(s-t_1^i)}e^{-\mu(t-s)}ds},
$$

and firm j's payoff from withholding the solution until t_2^j is

$$
\frac{\begin{array}{c}\frac{(p_1+p_2)}{2}\int_{t_1^i}^t He^{-H(s-t_1^i)}e^{-\mu(t-s)}ds + e^{-H(t-t_1^i)}[(p_1+\frac{p_2}{2})e^{-H(t_2^j-t)}e^{-\mu(t_2^j-t)} \\ +(p_1+p_2)\int_t^{t_2^j}\mu e^{-H(s-t)}e^{-\mu(s-t)}ds + \frac{(p_1+p_2)}{2}\int_t^{t_2^j}He^{-H(s-t)}e^{-\mu(s-t)}ds]\end{array}}{e^{-H(t-t_1^i)} + \int_{t_1^i}^t He^{-H(s-t_1^i)}e^{-\mu(t-s)}ds}.
$$

Firm j should be indifferent between disclosing and withholding the solution at t_1^j:

$$\frac{\frac{(p_1+p_2)}{2}\int_{t_1^i}^{t_1^j} He^{-H(s-t_1^i)}e^{-\mu(t_l^j-s)}ds + e^{-H(t_l^j-t_1^i)}[(p_1+\frac{p_2}{2})e^{-H(t_2^j-t_1^j)}e^{-\mu(t_2^j-t_1^j)} +(p_1+p_2)\int_{t_1^j}^{t_2^j}\mu e^{-H(s-t_1^j)}e^{-\mu(s-t_1^j)}ds + \frac{(p_1+p_2)}{2}\int_{t_1^j}^{t_2^j} He^{-H(s-t_1^j)}e^{-\mu(s-t_1^j)}ds]}{e^{-H(t_1^j-t_1^i)}+\int_{t_1^i}^{t_1^j} He^{-H(s-t_1^i)}e^{-\mu(t_1^j-s)}ds}$$

$$= \frac{(p_1+\frac{p_2}{2})e^{-H(t_1^j-t_1^i)}+\frac{(p_1+p_2)}{2}\int_{t_1^i}^{t_1^j} He^{-H(s-t_1^i)}e^{-\mu(t_1^j-s)}ds}{e^{-H(t_1^j-t_1^i)}+\int_{t_1^i}^{t_1^j} He^{-H(s-t_1^i)}e^{-\mu(t_1^j-s)}ds}$$

$$\Rightarrow \begin{aligned} &e^{-H(t_1^j-t_1^i)}[(p_1+\tfrac{p_2}{2})e^{-H(t_2^j-t_1^j)}e^{-\mu(t_2^j-t_1^j)} + (p_1+p_2)\int_{t_1^j}^{t_2^j}\mu e^{-H(s-t_1^j)}e^{-\mu(s-t_1^j)}ds \\ &\quad +\tfrac{(p_1+p_2)}{2}\int_{t_1^j}^{t_2^j} He^{-H(s-t_1^j)}e^{-\mu(s-t_1^j)}ds] = (p_1+\tfrac{p_2}{2})e^{-H(t_1^j-t_1^i)}\end{aligned}$$

$$\Rightarrow \frac{-\frac{1}{2}H}{\mu+H}p_1 + \frac{\frac{1}{2}\mu}{\mu+H}p_2 + p_1\frac{\frac{1}{2}H}{\mu+H}e^{-(\mu+H)(t_2^j-t_1^j)} + p_2\frac{-\frac{1}{2}\mu}{\mu+H}e^{-(\mu+H)(t_2^j-t_1^j)} = 0$$

$$\Rightarrow Hp_1 - \mu p_2 = 0,$$

which contradicts our assumption $p_1H > p_2\mu$.

Thus, firm j must also exit at the end of its first withhold region, $[t_1^j, t_2^j)$. If $t_1^j = t_2^j$, this withhold region has length zero, i.e., firm j will exit after its disclose region if it has not solved stage 1.

Appendix B. Proof of Theorem 1

From Lemma 3, we know that every equilibrium is a disclose-withhold-exit equilibrium. Now, we characterize the unique cutoffs t_1, t_2 and prove that the corresponding strategy profile is an equilibrium. The proof consists of five parts.

Part I: characterize t_1 **and** t_2**.** We first show that every equilibrium must be symmetric, i.e., $t_1^i = t_1^j = t_1$ and $t_2^i = t_2^j = t_2$.

Suppose that this is not the case, and suppose that firm i is the first firm to enter the withhold region at t_1^i. In the disclose region, the payoff from disclosure is $p_1 + \frac{p_2}{2}$; moreover, the expected payoff from withholding the solution from t_1^i onwards is

$$\begin{aligned} &e^{-H(t_2^j-t_1^i)}e^{-\mu(t_2^j-t_1^i)}(p_1+p_2) + \int_{t_1^i}^{t_2^j}\mu e^{-\mu(s-t_1^i)}e^{-H(s-t_1^i)}ds(p_1+p_2) \\ &+\int_{t_1^i}^{t_2^j} He^{-H(s-t_1^i)}e^{-\mu(s-t_1^i)}ds\frac{(p_1+p_2)}{2} \\ =& [-\frac{\mu}{H+\mu}e^{-H(s-t_1^i)-\mu(s-t_1^i)}|_{t_1^i}^{t_2^j} + e^{-H(t_2^j-t_1^i)-\mu(t_2^j-t_1^i)}](p_1+p_2) \\ &+[-\frac{H}{H+\mu}e^{-H(s-t_1^i)-\mu(s-t_1^i)}|_{t_1^i}^{t_2^j}]\frac{(p_1+p_2)}{2} \\ =& \frac{(p_1+p_2)}{2}[\frac{H}{H+\mu}e^{-(H+\mu)(t_2^j-t_1^i)} + \frac{H+2\mu}{H+\mu}].\end{aligned}$$

At cutoff t_1^i, if $t_1^i > 0$, firm i should be indifferent between disclosing and withholding the solution:

$$\frac{(p_1+p_2)}{2}[\frac{H}{H+\mu}e^{-(H+\mu)(t_2^j-t_1^i)} + \frac{H+2\mu}{H+\mu}] = p_1 + \frac{p_2}{2}$$

$$t_2^j - t_1^i = \frac{\ln\frac{\frac{p_1+\frac{p_2}{2}}{\frac{p_1+p_2}{2}} - \frac{H+2\mu}{H+\mu}}{\frac{H}{H+\mu}}}{-(H+\mu)}$$

$$t_2^j - t_1^i = \frac{\ln\frac{H(p_1+p_2)}{p_1H-p_2\mu}}{H+\mu}.$$

From firm j's perspective, at t_1^j, the payoff from disclosure is

$$\frac{(p_1+\frac{p_2}{2})e^{-H(t_1^j-t_1^i)} + \frac{(p_1+p_2)}{2}\int_{t_1^i}^{t_1^j} He^{-H(s-t_1^i)}e^{-\mu(t_1^j-s)}ds}{e^{-H(t_1^j-t_1^i)} + \int_{t_1^i}^{t_1^j} He^{-H(s-t_1^i)}e^{-\mu(t_1^j-s)}ds}.$$

Moreover, the expected payoff from withholding the solution from t_1^j onwards is

$$\frac{\begin{array}{c}\frac{(p_1+p_2)}{2}\int_{t_1^i}^{t_1^j} He^{-H(s-t_1^i)}e^{-\mu(t_1^j-s)}ds + e^{-H(t_1^j-t_1^i)}[(p_1+p_2)e^{-H(t_2^i-t_1^j)}e^{-\mu(t_2^i-t_1^j)} \\ +(p_1+p_2)\int_{t_1^j}^{t_2^i}\mu e^{-H(s-t_1^j)}e^{-\mu(s-t_1^j)}ds + \frac{(p_1+p_2)}{2}\int_{t_1^j}^{t_2^i} He^{-H(s-t_1^j)}e^{-\mu(s-t_1^j)}ds]\end{array}}{e^{-H(t_1^j-t_1^i)} + \int_{t_1^i}^{t_1^j} He^{-H(s-t_1^i)}e^{-\mu(t_1^j-s)}ds}.$$

At cutoff t_1^j, if $t_1^j < t_2^j$, firm j should be indifferent between disclosing and withholding the solution:

$$\begin{aligned}
&\frac{\begin{array}{c}\frac{(p_1+p_2)}{2}\int_{t_1^i}^{t_1^j} He^{-H(s-t_1^i)}e^{-\mu(t_1^j-s)}ds + e^{-H(t_1^j-t_1^i)}[(p_1+p_2)e^{-H(t_2^i-t_1^j)}e^{-\mu(t_2^i-t_1^j)} \\ +(p_1+p_2)\int_{t_1^j}^{t_2^i}\mu e^{-H(s-t_1^j)}e^{-\mu(s-t_1^j)}ds + \frac{(p_1+p_2)}{2}\int_{t_1^j}^{t_2^i} He^{-H(s-t_1^j)}e^{-\mu(s-t_1^j)}ds]\end{array}}{e^{-H(t_1^j-t_1^i)}+\int_{t_1^i}^{t_1^j} He^{-H(s-t_1^i)}e^{-\mu(t_1^j-s)}ds} \\
&\qquad = \frac{(p_1+\frac{p_2}{2})e^{-H(t_1^j-t_1^i)}+\frac{(p_1+p_2)}{2}\int_{t_1^i}^{t_1^j} He^{-H(s-t_1^i)}e^{-\mu(t_1^j-s)}ds}{e^{-H(t_1^j-t_1^i)}+\int_{t_1^i}^{t_1^j} He^{-H(s-t_1^i)}e^{-\mu(t_1^j-s)}ds} \\
&\Rightarrow \begin{array}{c} e^{-H(t_1^j-t_1^i)}[(p_1+p_2)e^{-H(t_2^i-t_1^j)}e^{-\mu(t_2^i-t_1^j)} + (p_1+p_2)\int_{t_1^j}^{t_2^i}\mu e^{-H(s-t_1^j)}e^{-\mu(s-t_1^j)}ds \\ +\frac{(p_1+p_2)}{2}\int_{t_1^j}^{t_2^i} He^{-H(s-t_1^j)}e^{-\mu(s-t_1^j)}ds] = (p_1+\frac{p_2}{2})e^{-H(t_1^j-t_1^i)}\end{array} \\
&\Rightarrow \frac{-\frac{1}{2}H}{\mu+H}p_1 + \frac{\frac{1}{2}\mu}{\mu+H}p_2 + p_1\frac{\frac{1}{2}H}{\mu+H}e^{-(\mu+H)(t_2^i-t_1^j)} + p_2\frac{\frac{1}{2}H}{\mu+H}e^{-(\mu+H)(t_2^i-t_1^j)} = 0 \\
&\Rightarrow \frac{\frac{1}{2}H}{\mu+H}(p_1+p_2)e^{-(\mu+H)(t_2^i-t_1^j)} = \frac{\frac{1}{2}H}{\mu+H}p_1 - \frac{\frac{1}{2}\mu}{\mu+H}p_2 \\
&\Rightarrow t_2^i - t_1^j = \frac{\ln\frac{H(p_1+p_2)}{p_1H-p_2\mu}}{H+\mu}.
\end{aligned}$$

Thus, we have

$$t_2^i - t_1^j = t_2^j - t_1^i = \frac{\ln\frac{H(p_1+p_2)}{p_1H-p_2\mu}}{H+\mu}.$$

The next step is to locate t_2^i and t_2^j. We equate firm i's instantaneous payoff rate at t with c:

$$c = \tilde{\lambda}^i(t)H\frac{(p_1+p_2)e^{-H(t_2^j-t_1^j)} + \frac{(p_1+p_2)}{2}\int_0^{t_2^j-t_1^j} He^{-Hs}e^{-(t-t_1^j-s)\mu}ds}{e^{-H(t_2^j-t_1^j)} + \int_0^{t_2^j-t_1^j} He^{-Hs}e^{-(t-t_1^j-s)\mu}ds},$$

where

$$\tilde{\lambda}^i(t) = \frac{\alpha e^{-2Ht_1^j-H(t-t_1^j)}[e^{-H(t_2^j-t_1^j)} + \int_0^{t_2^j-t_1^j} He^{-Hs}e^{-(t-t_1^j-s)\mu}ds]}{\alpha e^{-2Ht_1^j-H(t-t_1^j)}[e^{-H(t_2^j-t_1^j)} + \int_0^{t_2^j-t_1^j} He^{-Hs}e^{-(t-t_1^j-s)\mu}ds] + 1-\alpha}.$$

Thus,

$$c = H\frac{e^{-2Ht_1^j-H(t-t_1^j)}[(p_1+p_2)e^{-H(t_2^j-t_1^j)} + \frac{(p_1+p_2)}{2}\int_0^{t_2^j-t_1^j} He^{-Hs}e^{-(t-t_1^j-s)\mu}ds]}{e^{-2Ht_1^j-H(t-t_1^j)}[e^{-H(t_2^j-t_1^j)} + \int_0^{t_2^j-t_1^j} He^{-Hs}e^{-(t-t_1^j-s)\mu}ds] + \frac{1-\alpha}{\alpha}}.$$

In every equilibrium, this equality must hold at $t = t_2^i$:

$$\begin{aligned}
c &= H\frac{e^{-2Ht_1^j-H(t_2^i-t_1^j)}[(p_1+p_2)e^{-H(t_2^j-t_1^j)} + \frac{(p_1+p_2)}{2}\int_0^{t_2^j-t_1^j} He^{-Hs}e^{-(t_2^i-t_1^j-s)\mu}ds]}{e^{-2Ht_1^j-H(t_2^i-t_1^j)}[e^{-H(t_2^j-t_1^j)} + \int_0^{t_2^j-t_1^j} He^{-Hs}e^{-(t_2^i-t_1^j-s)\mu}ds] + \frac{1-\alpha}{\alpha}} \\
&= H\frac{e^{-Ht_2^j-Ht_2^i}(p_1+p_2) + e^{-Ht_1^j-Ht_2^i}\frac{(p_1+p_2)}{2}\int_0^{t_2^j-t_1^j} He^{-Hs}e^{-(t_2^i-t_1^j-s)\mu}ds}{e^{-Ht_2^j-Ht_2^i} + e^{-Ht_1^j-Ht_2^i}\int_0^{t_2^j-t_1^j} He^{-Hs}e^{-(t_2^i-t_1^j-s)\mu}ds + \frac{1-\alpha}{\alpha}}.
\end{aligned}$$

We call this equation the breakeven condition for firm i.

Similarly for firm j, we equate its instantaneous payoff rate at t with c:

$$c = \tilde{\lambda}^j(t)H\frac{\begin{array}{c}e^{-H(t-t_1^i)}[\left(\frac{\mu}{H+\mu} + \frac{H}{H+\mu}e^{-(H+\mu)(t_2^i-t)}\right)(p_1+p_2)\\ +\frac{H}{H+\mu}(1-e^{-(H+\mu)(t_2^i-t)})\frac{p_1+p_2}{2}] + \frac{(p_1+p_2)}{2}\int_0^{t-t_1^i} He^{-Hs}e^{-(t-t_1^i-s)\mu}ds\end{array}}{e^{-H(t-t_1^i)} + \int_0^{t-t_1^i} He^{-Hs}e^{-(t-t_1^i-s)\mu}ds},$$

where

$$\tilde{\lambda}^j(t) = \frac{\alpha e^{-2Ht_1^i-H(t-t_1^i)}[e^{-H(t-t_1^i)} + \int_0^{t-t_1^i} He^{-Hs}e^{-(t-t_1^i-s)\mu}ds]}{\alpha e^{-2Ht_1^i-H(t-t_1^i)}[e^{-H(t-t_1^i)} + \int_0^{t-t_1^i} He^{-Hs}e^{-(t-t_1^i-s)\mu}ds] + 1 - \alpha}.$$

Thus,

$$c = H\frac{\begin{array}{c}e^{-2Ht_1^i-H(t-t_1^i)}[\frac{p_1+p_2}{2}e^{-H(t-t_1^i)}(1 + \frac{\mu}{H+\mu} + \frac{H}{H+\mu}e^{-(H+\mu)(t_2^i-t)})\\ +\frac{(p_1+p_2)}{2}\int_0^{t-t_1^i} He^{-Hs}e^{-(t-t_1^i-s)\mu}ds]\end{array}}{e^{-2Ht_1^i-H(t-t_1^i)}[e^{-H(t-t_1^i)} + \int_0^{t-t_1^i} He^{-Hs}e^{-(t-t_1^i-s)\mu}ds] + \frac{1-\alpha}{\alpha}}.$$

Hence, firm j's breakeven condition is

$$c = H\frac{\begin{array}{c}e^{-2Ht_2^j}\frac{p_1+p_2}{2}(1 + \frac{\mu}{H+\mu} + \frac{H}{H+\mu}e^{-(H+\mu)(t_2^i-t_2^j)})\\ +e^{-Ht_1^i-Ht_2^j}\frac{p_1+p_2}{2}\int_0^{t_2^j-t_1^i} He^{-Hs}e^{-(t_2^j-t_1^i-s)\mu}ds\end{array}}{e^{-2Ht_2^j} + e^{-Ht_1^i-Ht_2^j}\int_0^{t_2^j-t_1^i} He^{-Hs}e^{-(t_2^j-t_1^i-s)\mu}ds + \frac{1-\alpha}{\alpha}}.$$

In summary,

$$\begin{cases} c = H\frac{e^{-Ht_2^j-Ht_2^i}(p_1+p_2)+e^{-Ht_1^j-Ht_2^i}\frac{(p_1+p_2)}{2}\int_0^{t_2^j-t_1^j} He^{-Hs}e^{-(t_2^i-t_1^j-s)\mu}ds}{e^{-Ht_2^j-Ht_2^i}+e^{-Ht_1^j-Ht_2^i}\int_0^{t_2^j-t_1^j} He^{-Hs}e^{-(t_2^i-t_1^j-s)\mu}ds+\frac{1-\alpha}{\alpha}} \\ c = H\frac{\begin{array}{c}e^{-2Ht_2^j}\frac{p_1+p_2}{2}(1 + \frac{\mu}{H+\mu} + \frac{H}{H+\mu}e^{-(H+\mu)(t_2^i-t_2^j)})\\ +e^{-Ht_1^i-Ht_2^j}\frac{p_1+p_2}{2}\int_0^{t_2^j-t_1^i} He^{-Hs}e^{-(t_2^j-t_1^i-s)\mu}ds\end{array}}{e^{-2Ht_2^j}+e^{-Ht_1^i-Ht_2^j}\int_0^{t_2^j-t_1^i} He^{-Hs}e^{-(t_2^j-t_1^i-s)\mu}ds+\frac{1-\alpha}{\alpha}}. \end{cases} \tag{5}$$

Let $\Delta = H(t_2^i - t_2^j) > 0$, $x = e^{-Ht_2^j-Ht_2^i} > 0$, $y = e^{-Ht_1^j-Ht_2^i}\int_0^{t_2^j-t_1^j} He^{-Hs}e^{-(t_2^i-t_1^j-s)\mu}ds > 0$, $z = e^{H(t_1^j-t_1^i)}\frac{\int_0^{t_2^j-t_1^i} He^{-Hs}e^{-(t_2^j-t_1^i-s)\mu}ds}{\int_0^{t_2^j-t_1^j} He^{-Hs}e^{-(t_2^i-t_1^j-s)\mu}ds} > 1$, and $w = 1 + \frac{\mu}{H+\mu} + \frac{H}{H+\mu}e^{-(H+\mu)(t_2^i-t_2^j)} = 1 + \frac{\mu}{H+\mu} + \frac{H}{H+\mu}e^{-\frac{H+\mu}{H}\Delta} \in (1, 2)$. We can rewrite (5) as

$$c = H\frac{(p_1+p_2)x + \frac{(p_1+p_2)}{2}y}{x + y + \frac{1-\alpha}{\alpha}} = H\frac{\frac{p_1+p_2}{2}xe^{\Delta}w + \frac{(p_1+p_2)}{2}ye^{\Delta}z}{xe^{\Delta} + ye^{\Delta}z + \frac{1-\alpha}{\alpha}}.$$

Let $d^i = H\frac{(p_1+p_2)x+\frac{(p_1+p_2)}{2}y}{x+y+\frac{1-\alpha}{\alpha}}$ and $d^j = H\frac{\frac{p_1+p_2}{2}xe^{\Delta}w+\frac{(p_1+p_2)}{2}ye^{\Delta}z}{xe^{\Delta}+ye^{\Delta}z+\frac{1-\alpha}{\alpha}}$. Fixing the other parameters, we know that

$$\begin{aligned} & d^i < d^j \\ \Leftrightarrow\ & (2x+y)(xe^{\Delta} + ye^{\Delta}z + \frac{1-\alpha}{\alpha}) < (x+y+\frac{1-\alpha}{\alpha})(xe^{\Delta}w + ye^{\Delta}z) \\ \Leftrightarrow\ & (2-w)x^2e^{\Delta} + (z+1-w)xye^{\Delta} < (e^{\Delta}w-2)\frac{1-\alpha}{\alpha}x + (e^{\Delta}z-1)\frac{1-\alpha}{\alpha}y \\ \Leftarrow\ & (2-w)xe^{\Delta} < (e^{\Delta}w-2)\frac{1-\alpha}{\alpha} \text{ and } (z+1-w)xe^{\Delta} < (e^{\Delta}z-1)\frac{1-\alpha}{\alpha} \\ \Leftrightarrow\ & (2-w)e^{-2Ht_2^j} < (e^{\Delta}w-2)\frac{1-\alpha}{\alpha} \text{ and } (z+1-w)e^{-2Ht_2^j} < (e^{\Delta}z-1)\frac{1-\alpha}{\alpha} \\ \Leftarrow\ & e^{-2Ht_2^j} \le \frac{1-\alpha}{\alpha} \text{ and } 2-w < e^{\Delta}w-2 \text{ and } z+1-w < e^{\Delta}z-1. \end{aligned}$$

We first prove that $e^{-2Ht_2^j} \le \frac{1-\alpha}{\alpha}$. When $\alpha < \frac{1}{2}$, this is clearly true. When $\alpha \ge \frac{1}{2}$, we consider a hypothetical case of $t_2^i = \infty$. Rewrite the breakeven condition for firm j in this case:

$$
\begin{aligned}
c &= \tilde{\lambda}^j(t_2^j)H\frac{p_1+p_2}{2}\frac{e^{-H(t_2^j-t_1^i)}(1+\frac{\mu}{H+\mu})+\int_0^{t_2^j-t_1^i}He^{-Hs}e^{-(t_2^j-t_1^i-s)\mu}ds}{e^{-H(t_2^j-t_1^i)}+\int_0^{t_2^j-t_1^i}He^{-Hs}e^{-(t_2^j-t_1^i-s)\mu}ds}\\
&= \tilde{\lambda}^j(t_2^j)H\frac{p_1+p_2}{2}(1+\frac{\mu}{H+\mu}\frac{1}{1+\frac{H}{H-\mu}(e^{(H-\mu)(t_2^j-t_1^i)}-1)}), \qquad (6)\\
&\text{s.t. } t_1^i = t_2^j - \frac{\ln\frac{H(p_1+p_2)}{p_1H-p_2\mu}}{H+\mu} \text{ if } t_2^j \geq \frac{\ln\frac{H(p_1+p_2)}{p_1H-p_2\mu}}{H+\mu};\ t_1^i = 0 \text{ otherwise.}
\end{aligned}
$$

Clearly, $\tilde{\lambda}^j(t_2^j)$ is decreasing in t_2^j. To prove that the right-hand side of (6) is decreasing in t_2^j, it then suffices to prove that $\frac{H}{\mu-H}[1-e^{(H-\mu)t}]$ is increasing in t. If $H>\mu$, $1-e^{(H-\mu)t}<0$ and $\mu-H<0$. As t increases, $1-e^{(H-\mu)t}$ decreases; $\frac{H}{\mu-H}[1-e^{(H-\mu)t}]$ increases. If $H<\mu$, $1-e^{(H-\mu)t}>0$ and $\mu-H>0$. As t increases, $1-e^{(H-\mu)t}$ increases; $\frac{H}{\mu-H}[1-e^{(H-\mu)t}]$ increases. Hence, (6) yields a unique solution for t_2^j.

Next, we claim that every t_2^j thus derived satisfies $e^{-2Ht_2^j}\leq\frac{1-\alpha}{\alpha}$. From the breakeven condition (6), we have

$$
\begin{aligned}
c =& H\frac{p_1+p_2}{2}\frac{\frac{\mu}{H+\mu}e^{-H(t_2^j-t_1^i)}+\frac{H}{H-\mu}e^{-\mu(t_2^j-t_1^i)}-\frac{\mu}{H-\mu}e^{-H(t_2^j-t_1^i)}}{\frac{H}{H-\mu}e^{-\mu(t_2^j-t_1^i)}-\frac{\mu}{H-\mu}e^{-H(t_2^j-t_1^i)}+\frac{1-\alpha}{\alpha}e^{Ht_1^i+Ht_2^j}}\\
e^{Ht_1^i+Ht_2^j} =& \frac{\alpha}{1-\alpha}\Big(\frac{H\frac{p_1+p_2}{2}[\frac{\mu}{H+\mu}e^{-H(t_2^j-t_1^i)}+\frac{H}{H-\mu}e^{-\mu(t_2^j-t_1^i)}-\frac{\mu}{H-\mu}e^{-H(t_2^j-t_1^i)}]}{c}\\
&+\frac{\mu}{H-\mu}e^{-H(t_2^j-t_1^i)}-\frac{H}{H-\mu}e^{-\mu(t_2^j-t_1^i)}\Big)\\
e^{2Ht_1^i} =& \frac{\alpha}{1-\alpha}\Big(\frac{H\frac{p_1+p_2}{2}[\frac{\mu}{H+\mu}e^{-H(t_2^j-t_1^i)}+\frac{H}{H-\mu}e^{-\mu(t_2^j-t_1^i)}-\frac{\mu}{H-\mu}e^{-H(t_2^j-t_1^i)}]}{c}\\
&+\frac{\mu}{H-\mu}e^{-H(t_2^j-t_1^i)}-\frac{H}{H-\mu}e^{-\mu(t_2^j-t_1^i)}\Big)e^{H(t_1^i-t_2^j)}.
\end{aligned}
$$

Note that $t_2^j-t_1^i$ is a constant when $t_2^j\geq\frac{\ln\frac{H(p_1+p_2)}{p_1H-p_2\mu}}{H+\mu}$, and by assumption A3, we know that $t_2^j\geq\frac{\ln\frac{H(p_1+p_2)}{p_1H-p_2\mu}}{H+\mu}$, for every $\alpha\geq\frac{1}{2}$. Hence,

$$
\begin{aligned}
&\Big(\frac{H\frac{p_1+p_2}{2}\Big[\frac{\mu}{H+\mu}e^{-H(t_2^j-t_1^i)}+\frac{H}{H-\mu}e^{-\mu(t_2^j-t_1^i)}-\frac{\mu}{H-\mu}e^{-H(t_2^j-t_1^i)}\Big]}{c}\\
&\quad+\frac{\mu}{H-\mu}e^{-H(t_2^j-t_1^i)}-\frac{H}{H-\mu}e^{-\mu(t_2^j-t_1^i)}\Big)e^{H(t_1^i-t_2^j)}\geq 1\\
&e^{2Ht_1^i}\geq\frac{\alpha}{1-\alpha}\\
&e^{-2Ht_1^i}\leq\frac{1-\alpha}{\alpha}.
\end{aligned}
$$

Therefore, $e^{-2Ht_2^j}\leq e^{-2Ht_1^i}\leq\frac{1-\alpha}{\alpha}$.

Now consider any $t_2^i<\infty$ and any corresponding t_2^j satisfying firm j's original breakeven condition in (5). If t_2^i were to become ∞, firm j's instantaneous net payoff rate would become negative, and by our previous analysis, t_2^j must shift to the left for the breakeven condition to hold. Hence, the original t_2^j must also satisfy $e^{-2Ht_2^j}\leq\frac{1-\alpha}{\alpha}$.

Next, note that $2-w=e^{\Delta}w-2$ when $\Delta=0$ and that

$$
\begin{aligned}
\frac{d(e^{\Delta}w-2-(2-w))}{de^{\Delta}} =& 1+\frac{\mu}{H+\mu}+\frac{H}{H+\mu}(-\frac{\mu}{H})(e^{\Delta})^{-\frac{\mu}{H}-1}\\
&+\frac{H}{H+\mu}(-\frac{H+\mu}{H})(e^{\Delta})^{-\frac{H+\mu}{H}-1}\\
\geq& 0
\end{aligned}
$$

when $\Delta>0$; hence, $2-w<e^{\Delta}w-2$.

Finally, observe that $e^{\Delta}z-1-(z+1-w)\geq e^{\Delta}+w-3$ and that $e^{\Delta}+w-3=0$ when $\Delta=0$; moreover, we have

$$
\frac{d(e^{\Delta}+w-3)}{de^{\Delta}}=1+\frac{H}{H+\mu}(-\frac{H+\mu}{H})(e^{\Delta})^{-\frac{H+\mu}{H}-1}>0.
$$

Hence, $z+1-w<e^{\Delta}z-1$.

Therefore, we have $c = d^i < d^j = c$, a contradiction. We can then conclude that every equilibrium strategy profile must be symmetric. Henceforth, we use t_1 and t_2 to denote the symmetric equilibrium cutoffs of time.

Now, we characterize t_1 and t_2. As calculated before, we first have a necessary equilibrium condition:

$$t_1 = t_2 - \frac{\ln \frac{H(p_1+p_2)}{p_1H - p_2\mu}}{H+\mu} \text{ if } t_2 \geq \frac{\ln \frac{H(p_1+p_2)}{p_1H - p_2\mu}}{H+\mu};\ t_1 = 0 \text{ otherwise.}$$

Thus, given any t_2, we can always find a unique t_1. In the case of $p_1H \leq p_2\mu$, since $t_2 \geq \frac{\ln \frac{H(p_1+p_2)}{p_1H-p_2\mu}}{H+\mu}$ can never hold, t_1 is always equal to 0.

The next step is to show that t_2 is also unique. We have a breakeven condition by Bayesian updating:

$$c \ = \ \tilde{\lambda}(t_2)H\frac{(p_1+p_2)e^{-H(t_2-t_1)} + \frac{(p_1+p_2)}{2}\int_0^{t_2-t_1} He^{-Hs}e^{-(t_2-t_1-s)\mu}ds}{e^{-H(t_2-t_1)} + \int_0^{t_2-t_1} He^{-Hs}e^{-(t_2-t_1-s)\mu}ds} \tag{7}$$

$$\text{s.t. } t_1 = t_2 - \frac{\ln \frac{H(p_1+p_2)}{p_1H - p_2\mu}}{H+\mu} \text{ if } t_2 \geq \frac{\ln \frac{H(p_1+p_2)}{p_1H - p_2\mu}}{H+\mu};\ \ t_1 = 0 \text{ otherwise.}$$

We already know that the right-hand side of (7) is decreasing in t, which implies a unique solution to t_2.

Now that we have unique characterization of t_1 and t_2, we prove that the corresponding strategy profile is an equilibrium in the next four parts. Following the definition of an equilibrium, if a strategy of a firm is its strict best response from time t' onwards, the firm will (expect itself to) use this strategy for $t \geq t'$ in every best response at time $t' < t$. Hence, we proceed by checking each firm's incentive backwards ($t \geq t_2$, $t \in [t_1, t_2)$, $t < t_1$).

Part II: prove that a firm will exit at t_2. In principle, this requires the calculation of a firm's expected payoff from remaining for an arbitrary length of time after t_2 and showing that this payoff is always below the expected cost, which involves the explicit characterization of the distribution of the game's ending time and may not be tractable. Instead, we find that with assumption A3, it suffices to focus on a much simpler problem: we examine a firm's instantaneous rate of payoff and find that it is always below the cost per unit of time c after t_2, which renders it non-profitable to remain in the game any longer.

We evaluate the firm's incentive at time $t_2 + \Delta t$. Remaining in the game yields an instantaneous rate of expected payoff:

$$\begin{aligned}
&\tilde{\lambda}(t_2+\Delta t)H\frac{(p_1+p_2)e^{-H(t_2-t_1)} + \frac{(p_1+p_2)}{2}\int_0^{t_2-t_1} He^{-Hs}e^{-(t_2+\Delta t-t_1-s)\mu}ds}{e^{-H(t_2-t_1)} + \int_0^{t_2-t_1} He^{-Hs}e^{-(t_2+\Delta t-t_1-s)\mu}ds} \\
&= \alpha H\frac{e^{-2Ht_1-H(t_2+\Delta t-t_1)}[(p_1+p_2)e^{-H(t_2-t_1)} + \frac{(p_1+p_2)}{2}\int_0^{t_2-t_1} He^{-Hs}e^{-(t_2+\Delta t-t_1-s)\mu}ds]}{\alpha e^{-2Ht_1-H(t_2+\Delta t-t_1)}[e^{-H(t_2-t_1)} + \int_0^{t_2-t_1} He^{-Hs}e^{-(t_2+\Delta t-t_1-s)\mu}ds] + 1-\alpha} \\
&= H\frac{(p_1+p_2)}{2} \\
&\quad \left\{1 + \frac{e^{-2Ht_1-H(t_2+\Delta t-t_1)}e^{-H(t_2-t_1)} - \frac{1-\alpha}{\alpha}}{e^{-2Ht_1-H(t_2+\Delta t-t_1)}\{e^{-H(t_2-t_1)} + \frac{H}{\mu-H}[e^{-H(t_2-t_1)} - e^{-\mu(t_2-t_1)}]e^{-\Delta t\mu}\} + \frac{1-\alpha}{\alpha}}\right\}.
\end{aligned} \tag{8}$$

At $\Delta t = 0$, (7)=(8), and thus, the instantaneous payoff rate is continuous at t_2. By a similar argument to that in Part I, $e^{-2Ht_1} \leq \frac{1-\alpha}{\alpha}$ under assumption A3. Hence,

$$e^{-2Ht_1-H(t_2+\Delta t-t_1)}e^{-H(t_2-t_1)} - \frac{1-\alpha}{\alpha} = e^{-2Ht_2-H\Delta t} - \frac{1-\alpha}{\alpha} \leq e^{-2Ht_1} - \frac{1-\alpha}{\alpha} \leq 0$$

for every $\alpha \in (0,1)$. Moreover, the above expression is decreasing in Δt. However, the term

$$e^{-2Ht_1-H(t_2+\Delta t-t_1)}\{e^{-H(t_2-t_1)} + \frac{H}{\mu-H}[e^{-H(t_2-t_1)} - e^{-\mu(t_2-t_1)}]e^{-\Delta t\mu}\}$$

is positive and decreasing in Δt, and thus, (8) decreases in Δt. As the breakeven condition holds at $\Delta t = 0$ (i.e., $t = t_2$), the instantaneous payoff rate is smaller than c for all $\Delta t > 0$.

Consider the following deviation by a firm: it will remain in the game until $t_3 > t_2$, i.e., it will exit the competition if it has not solved stage 1 and the opponent has not solved stage 2 by t_3. Note that disclosing the solution to stage 1 after t_2 will not change the firm's payoff, and thus, we can focus on its strategy of withholding the solution until either the game ends or the firm exits. Let E denote the event that either the firm solves stage 1 or its opponent solves stage 2, and let $F(\Delta t)$ denote the probability that E occurs before $\Delta t \in [0, t_3 - t_2]$. From the model setting and our previous analysis, we know that $F(t)$ is differentiable. The firm's (net) expected payoff following this deviation, evaluated at t_2, is

$$\int_0^{t_3-t_2} (q(\Delta t) - c)dF(\Delta t) - (1 - F(t_3 - t_2))(t_3 - t_2)c$$

where $q(\Delta t)$ is the instantaneous payoff rate characterized in (8). Since $q(\Delta t) < c$ for all $\Delta t > 0$, we know that the above payoff is negative. Hence, it is never optimal for a firm to remain in the game after t_2.

Part III: prove that a firm will never exit before t_2. The above argument has proved that exit is optimal after t_2. For $t \in [0, t_1]$, the instantaneous payoff rate from remaining for another dt and disclosing the solution if stage 1 is solved is

$$\tilde{\lambda}(t)[H(p_1 + \frac{p_2}{2}) + H\frac{p_2}{2}] = \tilde{\lambda}(t)H(p_1 + p_2). \tag{9}$$

The right-hand side of (9) is greater than the right-hand side of (7). Hence we know that the payoff rate of staying in the game must be greater than c, and therefore it is never optimal for a firm to exit before t_1.

For $t \in (t_1, t_2]$, the instantaneous payoff rate from continuing research (and adopting a withhold strategy) before solving stage 1, rather than exiting immediately, is

$$\begin{aligned}&\frac{\begin{array}{l}\tilde{\lambda}(t)H\{\frac{p_1+p_2}{2}\int_{t_1}^{t} He^{-H(s-t_1)}e^{-\mu(t-s)}ds + e^{-H(t-t_1)}[(p_1+p_2)e^{-H(t_2-t)}e^{-\mu(t_2-t)}\\ \quad +(p_1+p_2)\int_t^{t_2}\mu e^{-\mu(s-t)}e^{-H(s-t)}ds + \frac{p_1+p_2}{2}\int_t^{t_2} He^{-H(s-t)}e^{-\mu(s-t)}ds]\}\end{array}}{\int_{t_1}^{t} He^{-H(s-t_1)}e^{-\mu(t-s)}ds + e^{-H(t-t_1)}}\\ &= \frac{\begin{array}{l}H\{\frac{p_1+p_2}{2}\int_{t_1}^{t} He^{-H(s-t_1)}e^{-\mu(t-s)}ds + e^{-H(t-t_1)}[(p_1+p_2)e^{-H(t_2-t)}e^{-\mu(t_2-t)}\\ \quad +(p_1+p_2)\int_t^{t_2}\mu e^{-\mu(s-t)}e^{-H(s-t)}ds + \frac{p_1+p_2}{2}\int_t^{t_2} He^{-H(s-t)}e^{-\mu(s-t)}ds]\}\end{array}}{\int_{t_1}^{t} He^{-H(s-t_1)}e^{-\mu(t-s)}ds + e^{-H(t-t_1)} + \frac{1-\alpha}{\alpha}e^{2Ht_1+H(t-t_1)}}.\end{aligned} \tag{10}$$

The numerator of (10) is

$$\begin{aligned}&H\{\frac{p_1+p_2}{2}\int_{t_1}^{t} He^{-H(s-t_1)}e^{-\mu(t-s)}ds + e^{-H(t-t_1)}[(p_1+p_2)e^{-H(t_2-t)}e^{-\mu(t_2-t)}\\ &+(p_1+p_2)\int_t^{t_2}\mu e^{-\mu(s-t)}e^{-H(s-t)}ds + \frac{p_1+p_2}{2}\int_t^{t_2} He^{-H(s-t)}e^{-\mu(s-t)}ds]\}\\ &= H\{\frac{p_1+p_2}{2}\frac{H}{-(H-\mu)}e^{Ht_1-\mu t}(e^{-(H-\mu)t} - e^{-(H-\mu)t_1}) + e^{-H(t-t_1)}[(p_1+p_2)e^{-H(t_2-t)}e^{-\mu(t_2-t)}\\ &+(p_1+p_2)\frac{\mu}{-(H+\mu)}e^{Ht+\mu t}(e^{-(H+\mu)t_2} - e^{-(H+\mu)t})\\ &+\frac{p_1+p_2}{2}\frac{H}{-(H+\mu)}e^{Ht+\mu t}(e^{-(H+\mu)t_2} - e^{-(H+\mu)t})]\}\\ &= H\{\frac{p_1+p_2}{2}(-\frac{H}{H-\mu})(e^{H(t_1-t)} - e^{\mu(t_1-t)}) + e^{-H(t-t_1)}[(p_1+p_2)e^{-H(t_2-t)}e^{-\mu(t_2-t)}\\ &+(p_1+p_2)(-\frac{\mu}{H+\mu})(e^{-(H+\mu)(t_2-t)} - 1) + \frac{p_1+p_2}{2}(-\frac{H}{H+\mu})(e^{-(H+\mu)(t_2-t)} - 1)]\}\\ &= H(p_1+p_2)\{(-\frac{\frac{1}{2}H}{H-\mu})(e^{H(t_1-t)} - e^{\mu(t_1-t)}) + e^{-H(t-t_1)}[\frac{\frac{1}{2}H}{H+\mu}e^{-(H+\mu)(t_2-t)} + \frac{\frac{1}{2}H+\mu}{H+\mu}]\}\\ &= H(p_1+p_2)e^{-H(t-t_1)}[-\frac{\frac{1}{2}H}{H-\mu} + \frac{\frac{1}{2}H}{H-\mu}e^{(H-\mu)(t-t_1)} + \frac{\frac{1}{2}H+\mu}{H+\mu} + \frac{\frac{1}{2}H}{H+\mu}e^{-(H+\mu)(t_2-t)}].\end{aligned} \tag{11}$$

Omitting the constant term $H(p_1+p_2)$, we focus on the following expression: $e^{-H(t-t_1)}[-\frac{\frac{1}{2}H}{H-\mu} + \frac{\frac{1}{2}H}{H-\mu}e^{(H-\mu)(t-t_1)} + \frac{\frac{1}{2}H+\mu}{H+\mu} + \frac{\frac{1}{2}H}{H+\mu}e^{-(H+\mu)(t_2-t)}]$. Take the first-order derivative with respect to t:

$$\begin{aligned}&\frac{\partial\{e^{-H(t-t_1)}[-\frac{\frac{1}{2}H}{H-\mu} + \frac{\frac{1}{2}H}{H-\mu}e^{(H-\mu)(t-t_1)} + \frac{\frac{1}{2}H+\mu}{H+\mu} + \frac{\frac{1}{2}H}{H+\mu}e^{-(H+\mu)(t_2-t)}]\}}{\partial t}\\ &= -He^{-H(t-t_1)}(-\frac{\frac{1}{2}H}{H-\mu} + \frac{\frac{1}{2}H+\mu}{H+\mu}) - \frac{\frac{1}{2}H\mu}{H-\mu}e^{-\mu(t-t_1)} + \frac{\frac{1}{2}H\mu}{H+\mu}e^{-H(t_2-t_1)-\mu(t_2-t)}\\ &= \frac{\frac{1}{2}H\mu}{H+\mu}[-e^{-H(t-t_1)} + e^{-H(t_2-t_1)-\mu(t_2-t)}] + \frac{\frac{1}{2}\mu H}{H-\mu}(e^{-H(t-t_1)} - e^{-\mu(t-t_1)}).\end{aligned} \tag{12}$$

The first term of (12) is smaller than 0 since $-H(t-t_1) > -H(t_2-t_1) - \mu(t_2-t)$. Furthermore, the second term is smaller than 0. Hence, the numerator of (10) is decreasing in t.

The denominator of (10) is

$$\begin{aligned}&\int_{t_1}^{t} He^{-H(s-t_1)}e^{-\mu(t-s)}ds + e^{-H(t-t_1)} + e^{2Ht_1+H(t-t_1)}\\ &= \frac{-\mu}{H-\mu}e^{H(t_1-t)} + \frac{H}{H-\mu}e^{\mu(t_1-t)} + \frac{1-\alpha}{\alpha}e^{2Ht_1+H(t-t_1)}.\end{aligned}$$

Take the first-order derivative with respect to t:

$$\frac{\partial[\frac{-\mu}{H-\mu}e^{H(t_1-t)}+\frac{H}{H-\mu}e^{\mu(t_1-t)}+\frac{1-\alpha}{\alpha}e^{2Ht_1+H(t-t_1)}]}{\partial t}$$
$$=\frac{H\mu}{H-\mu}[e^{-H(t-t_1)}-e^{-\mu(t-t_1)}]+\frac{1-\alpha}{\alpha}He^{2Ht_1+H(t-t_1)}. \tag{13}$$

The first-order derivative of $e^{-H(t-t_1)}-e^{-\mu(t-t_1)}$ with respect to t is

$$-He^{-H(t-t_1)}+\mu e^{-\mu(t-t_1)},$$

which is 0 at

$$He^{-H(t-t_1)}=\mu e^{-\mu(t-t_1)}.$$

Thus, the minimum value of $\frac{H\mu}{H-\mu}[e^{-H(t-t_1)}-e^{-\mu(t-t_1)}]$ is obtained when $He^{-H(t-t_1)}=\mu e^{-\mu(t-t_1)}$. Then, we have

$$\begin{aligned}&\frac{H\mu}{H-\mu}[e^{-H(t-t_1)}-e^{-\mu(t-t_1)}]+\frac{1-\alpha}{\alpha}He^{2Ht_1+H(t-t_1)}\\ \geq\ &\frac{1-\alpha}{\alpha}He^{2Ht_1+H(t-t_1)}+\frac{H\mu}{H-\mu}[e^{-H(t-t_1)}-\frac{H}{\mu}e^{-H(t-t_1)}]\\ =\ &\frac{1-\alpha}{\alpha}He^{2Ht_1+H(t-t_1)}+\frac{H\mu}{H-\mu}e^{-H(t-t_1)}[1-\frac{H}{\mu}]\\ =\ &\frac{1-\alpha}{\alpha}He^{2Ht_1+H(t-t_1)}-He^{-H(t-t_1)}\\ \geq\ &\frac{1-\alpha}{\alpha}He^{2Ht_1}-He^{-H(t-t_1)}.\end{aligned}$$

The above inequality is binding only when $H=\mu$ and $t=t_1=0$. Thus, to prove that the denominator of (10) is increasing in t, it suffices to prove that $\frac{1-\alpha}{\alpha}e^{2Ht_1}\geq 1$. When $\alpha<\frac{1}{2}$, it is clear that $\frac{1-\alpha}{\alpha}e^{2Ht_1}\geq 1$ since $t_1\geq 0$. When $\alpha\geq\frac{1}{2}$, from Part I, we know that $e^{2Ht_1}\geq\frac{\alpha}{1-\alpha}$, and hence, $\frac{1-\alpha}{\alpha}e^{2Ht_1}\geq 1$.

Therefore, the denominator of (10) is increasing in t. In conclusion, (10) is decreasing in t for $t\in[t_1,t_2]$. Notice that (10)$=c$ at $t=t_2$. Thus, for $t\in[t_1,t_2]$ the value of staying in the game is always greater than the cost, and the firm will not exit.

Part IV: prove that withholding the solution after solving stage 1 is optimal in the withhold region. At time $t\in[t_1,t_2)$, the expected payoff of disclosing is

$$\frac{(p_1+\frac{p_2}{2})e^{-H(t-t_1)}+\frac{p_1+p_2}{2}\int_{t_1}^{t}He^{-H(s-t_1)}e^{-\mu(t-s)}ds}{e^{-H(t-t_1)}+\int_{t_1}^{t}He^{-H(s-t_1)}e^{-\mu(t-s)}ds}. \tag{14}$$

The expected payoff of withholding the solution is

$$\frac{\begin{aligned}&\tfrac{p_1+p_2}{2}\int_{t_1}^{t}He^{-H(s-t_1)}e^{-\mu(t-s)}ds+e^{-H(t-t_1)}[(p_1+p_2)e^{-H(t_2-t)}e^{-\mu(t_2-t)}\\&+(p_1+p_2)\int_{t}^{t_2}\mu e^{-H(s-t)}e^{-\mu(s-t)}ds+\tfrac{p_1+p_2}{2}\int_{t}^{t_2}He^{-H(s-t)}e^{-\mu(s-t)}ds]\end{aligned}}{e^{-H(t-t_1)}+\int_{t_1}^{t}He^{-H(s-t_1)}e^{-\mu(t-s)}ds}. \tag{15}$$

Then, the difference between withholding and disclosing the solution is (15) and (14):

$$\begin{aligned}&\frac{\begin{aligned}&\tfrac{p_1+p_2}{2}\int_{t_1}^{t}He^{-H(s-t_1)}e^{-\mu(t-s)}ds+e^{-H(t-t_1)}[(p_1+p_2)e^{-H(t_2-t)}e^{-\mu(t_2-t)}\\&+(p_1+p_2)\int_{t}^{t_2}\mu e^{-H(s-t)}e^{-\mu(s-t)}ds+\tfrac{p_1+p_2}{2}\int_{t}^{t_2}He^{-H(s-t)}e^{-\mu(s-t)}ds]\\&-(p_1+\tfrac{p_2}{2})e^{-H(t-t_1)}-\tfrac{p_1+p_2}{2}\int_{t_1}^{t}He^{-H(s-t_1)}e^{-\mu(t-s)}ds\end{aligned}}{e^{-H(t-t_1)}+\int_{t_1}^{t}He^{-H(s-t_1)}e^{-\mu(t-s)}ds}\\ =\ &\frac{\begin{aligned}&e^{-H(t-t_1)}[(p_1+p_2)e^{-H(t_2-t)}e^{-\mu(t_2-t)}+(p_1+p_2)\int_{t}^{t_2}\mu e^{-H(s-t)}e^{-\mu(s-t)}ds\\&+\tfrac{p_1+p_2}{2}\int_{t}^{t_2}He^{-H(s-t)}e^{-\mu(s-t)}ds-(p_1+\tfrac{p_2}{2})]\end{aligned}}{e^{-H(t-t_1)}+\int_{t_1}^{t}He^{-H(s-t_1)}e^{-\mu(t-s)}ds}\\ =\ &\frac{e^{-H(t-t_1)}[(p_1+p_2)e^{-H(t_2-t)}e^{-\mu(t_2-t)}+(p_1+p_2)\frac{\mu+\frac{1}{2}H}{\mu+H}(1-e^{-(\mu+H)(t_2-t)})-(p_1+\frac{p_2}{2})]}{e^{-H(t-t_1)}+\int_{t_1}^{t}He^{-H(s-t_1)}e^{-\mu(t-s)}ds}\end{aligned}$$

$$= \frac{e^{-H(t-t_1)}[\frac{-\frac{1}{2}H}{\mu+H}p_1 + \frac{\frac{1}{2}\mu}{\mu+H}p_2 + (p_1+p_2)\frac{\frac{1}{2}H}{\mu+H}e^{-(\mu+H)(t_2-t)}]}{e^{-H(t-t_1)} - \frac{1}{H-\mu}[e^{H(t_1-t)} - e^{\mu(t_1-t)}]}$$

$$= \frac{\frac{-\frac{1}{2}H}{\mu+H}p_1 + \frac{\frac{1}{2}\mu}{\mu+H}p_2 + (p_1+p_2)\frac{\frac{1}{2}H}{\mu+H}e^{-(\mu+H)(t_2-t)}}{1 - \frac{1}{H-\mu}[1 - e^{(H-\mu)(t-t_1)}]}. \tag{16}$$

The numerator is increasing in t. The denominator is always positive. Notably, by our definition of equilibrium, (16) is zero at $t = t_1$. Then, as t increases from t_1, the numerator is always positive. Thus, (16) > 0 for $t \in [t_1, t_2)$. Hence, withholding the solution is confirmed to be superior to disclosure everywhere in the withhold region.

Part V: prove that disclosure is optimal after solving stage 1 in the disclose region. We discuss two types of deviation: type-1 deviation, where the firm will withhold the solution for some period and then disclose the solution, and type-2 deviation, where the firm will withhold the solution until it solves stage 2. We verify that both types of deviation are inferior to immediate disclosure. In the disclose region, i.e., $[0, t_1)$, the payoff from disclosure is $p_1 + \frac{1}{2}p_2$.

Suppose that the firm uses type-1 deviation. By the proof of Lemma 3, the payoff is always less than $p_1 + \frac{1}{2}p_2$, and thus, any type-1 deviation is not profitable.

Suppose that the firm uses type-2 deviation, and let U denote the expected payoff at time t_1; then, the expected payoff at time t is

$$e^{-H(t_1-t)}e^{-\mu(t_1-t)}U + \int_t^{t_1} \mu e^{-\mu(s-t)}e^{-H(s-t)}ds(p_1+p_2)$$
$$+ \int_t^{t_1} He^{-H(s-t)}e^{-\mu(s-t)}ds\frac{(p_1+p_2)}{2}. \tag{17}$$

The profit generated by the deviation is (17) minus the original payoff $p_1 + \frac{1}{2}p_2$. However, the firm should be indifferent between disclosing and withholding the solution at t_1; thus, $U = p_1 + \frac{1}{2}p_2$. This implies that this deviation is no different from a type-1 deviation where the firm resumes disclosure at t_1. Then, again by the proof of Lemma 3, any type-2 deviation is not profitable. Hence, disclosure is confirmed to be superior to withholding the solution everywhere in the disclose region. This completes the proof.

Appendix C. Proof of Proposition 1

It is straightforward that Lemmas 1 and 2 remain valid under NFRA. Furthermore, there must still be at least one withhold region in every equilibrium. We revise the proof of Lemma 3 here to show that every firm's equilibrium behavior still exhibits a disclose-withhold-exit pattern. To avoid redundant technical details, we assume a symmetric strategy profile (which can be proved to be the only possible equilibrium strategy profile using our argument in the proof of Theorem 1).

Suppose that there are multiple withhold regions, with the first being $[t_1, t_2)$. We analyze the firm's incentives at t_1.

For the first disclose region $[0, t_1)$, the payoff from disclosure is

$$p_1 + p_2\int_0^{+\infty} \mu e^{-\mu s}e^{-Hs}ds + \frac{p_2}{2}\int_0^{+\infty} \lambda e^{-\mu s}e^{-Hs}ds$$
$$= p_1 + \frac{\frac{1}{2}H+\mu}{H+\mu}p_2.$$

For the withhold region, the expected payoff from withholding from t_1 to t_2 is

$$e^{-H(t_2-t_1)}e^{-\mu(t_2-t_1)}(p_1 + \frac{\frac{1}{2}H+\mu}{H+\mu}p_2) + \int_{t_1}^{t_2} \mu e^{-\mu(s-t_1)}e^{-H(s-t_1)}ds \cdot (p_1+p_2)$$
$$+ \int_{t_1}^{t_2} He^{-H(s-t_1)}e^{-\mu(s-t_1)}ds \cdot \frac{(p_1+p_2)}{2}$$
$$= p_1[\frac{\frac{1}{2}H}{H+\mu}e^{-(H+\mu)(t_2-t_1)} + \frac{\frac{1}{2}H+\mu}{H+\mu}] + p_2\frac{\frac{1}{2}H+\mu}{H+\mu}.$$

At cutoff t_1, the difference between the expected payoff from withholding and disclosing the solution is

$$p_1[\frac{\frac{1}{2}H}{H+\mu}e^{-(H+\mu)(t_2-t_1)} + \frac{\frac{1}{2}H+\mu}{H+\mu}] + p_2\frac{\frac{1}{2}H+\mu}{H+\mu} - p_1 - \frac{\frac{1}{2}H+\mu}{H+\mu}p_2$$
$$= p_1\frac{\frac{1}{2}H}{H+\mu}[e^{-(H+\mu)(t_2-t_1)} - 1]$$

which is smaller than zero, contradicting our assumption that the firm weakly prefers withholding the solution at t_1.

Following the proof of Theorem 1, we characterize the unique cutoffs (t_1, t_2) and prove that the corresponding strategy profile is an equilibrium. The proof consists of five parts.

Part I: characterize $t_1$and t_2. In the disclose region, the payoff from disclosure is

$$p_1 + p_2\int_0^{+\infty} \mu e^{-\mu s}e^{-Hs}ds + \frac{p_2}{2}\int_0^{+\infty} \lambda e^{-\mu s}e^{-Hs}ds$$
$$= p_1 + \frac{\frac{1}{2}H+\mu}{H+\mu}p_2.$$

Similar to the proof of Theorem 1, the payoff from withholding the solution starting at t_1 is $\frac{(p_1+p_2)}{2}[\frac{H}{H+\mu}e^{-(H+\mu)(t_2-t_1)} + \frac{H+2\mu}{H+\mu}]$.

The firm should be indifferent between disclosing and withholding the solution at t_1:

$$\frac{p_1+p_2}{2}[\frac{H}{H+\mu}e^{-(H+\mu)(t_2-t_1)} + \frac{H+2\mu}{H+\mu}] = p_1 + \frac{\frac{1}{2}H+\mu}{H+\mu}p_2$$
$$(p_1+p_2)[\frac{\frac{1}{2}H}{H+\mu}e^{-(H+\mu)(t_2-t_1)} + \frac{\frac{1}{2}H+\mu}{H+\mu}] = p_1 + \frac{\frac{1}{2}H+\mu}{H+\mu}p_2$$
$$e^{-(H+\mu)(t_2-t_1)} = \frac{p_1}{p_1+p_2}$$
$$t_2-t_1 = \frac{\ln\left(\frac{p_1+p_2}{p_1}\right)}{H+\mu}.$$

Compare the above condition with the one that characterizes t_2-t_1 in the proof of Theorem 1, i.e., $t_2-t_1 = \frac{\ln\frac{H(p_1+p_2)}{p_1H-p_2\mu}}{H+\mu}$, t_2-t_1 is smaller under NFRA. Thus, the withhold region of our candidate equilibrium under NFRA is shorter than that in the baseline model.

The remainder of Part I is irrelevant to disclosure and directly follows the proof of Theorem 1.

Part II and **Part III** are irrelevant to disclosure and directly follow the proof of Theorem 1.

Part IV: prove that withholding the solution after solving stage 1 is optimal in the withhold region. At time $t_2 \in [t_1, t_2]$, the expected payoff from disclosure is

$$\frac{(p_1+\frac{\frac{1}{2}H+\mu}{H+\mu}p_2)e^{-H(t-t_1)} + \frac{p_1+p_2}{2}\int_{t_1}^{t} He^{-H(s-t_1)}e^{-\mu(t-s)}ds}{e^{-H(t-t_1)} + \int_{t_1}^{t} He^{-H(s-t_1)}e^{-\mu(t-s)}ds}. \tag{18}$$

The expected payoff from withholding the solution remains (15) in the proof of Theorem 1.

Then, the difference between withholding and disclosing the solution is (15)-(18):

$$\frac{\frac{-\frac{1}{2}H}{\mu+H}p_1 + (p_1+p_2)\frac{\frac{1}{2}H}{\mu+H}e^{-(\mu+H)(t_2-t)}}{1-\frac{1}{H-\mu}[1-e^{(H-\mu)(t-t_1)}]}. \tag{19}$$

The numerator is increasing in t. The denominator is always positive. By our definition of equilibrium, (19) is zero at $t = t_1$. Then, as t increases from t_1, the numerator is always positive. Thus, (19) > 0 for $t \in [t_1, t_2)$. Hence, withholding the solution is confirmed to be superior to disclosure everywhere in the withhold region.

Part V: prove that disclosure is optimal after solving stage 1 in the disclose region. In the disclose region, the payoff from disclosure is $p_1 + \frac{\frac{1}{2}H+\mu}{H+\mu}p_2$. The remainder of the proof follows the proof of Theorem 1 except that now $U = p_1 + \frac{\frac{1}{2}H+\mu}{H+\mu}p_2$.

Appendix D. Proof of Lemma 3′

By the proof of Lemma 3, there must exist at least one withhold region in every equilibrium. Suppose that, in a symmetric equilibrium, there is a series of withhold regions, $[t_1, t_2)$, $[t_3, t_4)$, ..., $[t_{2m-1}, t_{2m})$, $m \in \mathbb{N}^+$. We prove the following claim: at t_2, either firm must prefer to withhold the solution for an extra instant.

By assumption, a firm weakly prefers withholding the solution at t_1, which implies that

$$p_1 + \frac{p_2}{2}\int_t^{\infty} \mu e^{-(2\mu+r)(s-t)}ds$$
$$\le e^{-(H+\mu+r)(t_2-t_1)}(p_1 + \frac{\mu}{2\mu+r}p_2) + \int_{t_1}^{t_2} \mu e^{-(H+\mu+r)(s-t_1)}ds(p_1+p_2)$$
$$+ \int_{t_1}^{t_2} He^{-(H+\mu+r)(s-t_1)}\int_s^{t_2} 2\mu e^{-(2\mu+r)(u-s)}duds\frac{p_1+p_2}{2}$$
$$+ \int_{t_1}^{t_2} He^{-(H+\mu+r)(s-t_1)}e^{-(2\mu+r)(t_2-s)}ds\left(\frac{1}{2}p_1 + \frac{\mu}{2\mu+r}p_2\right).$$

At time t_2, which is the beginning of the opponent's disclose region, the payoff from disclosure stays unchanged. However, if the firm kept withholding the solution for a time length of $t_2 - t_1$, its payoff would be strictly larger than the right-hand side above since the opponent would disclose its solution if obtained between t_2 and t_3, which brings $\frac{p_1}{2}$ to each firm earlier in expectation. Therefore, the firm would strictly prefer withholding the solution for for a time length of $t_2 - t_1$ to disclosing it immediately, which is a contradiction.

Appendix E. Proof of Theorem 1′

Part I: characterize t_1 and t_2 . In the disclose region, the payoff from disclosure is $p_1 + \frac{\mu}{2\mu+r}p_2$. On the other hand, the expected payoff from withholding the solution from t_1 onwards is

$$\begin{aligned}
&e^{-(H+\mu+r)(t_2-t_1)}\int_{t_2}^{\infty}\mu e^{-(\mu+r)(s-t_2)}ds(p_1+p_2)+\int_{t_1}^{t_2}\mu e^{-(H+\mu+r)(s-t_1)}ds(p_1+p_2)\\
&+\int_{t_1}^{t_2}He^{-(H+\mu+r)(s-t_1)}\int_{s}^{\infty}2\mu e^{-2\mu(u-s)}e^{-r(u-s)}duds\frac{p_1+p_2}{2}\\
=\;&e^{-(H+\mu+r)(t_2-t_1)}\frac{\mu}{\mu+r}(p_1+p_2)+\frac{\mu}{H+\mu+r}[1-e^{-(H+\mu+r)(t_2-t_1)}](p_1+p_2)\\
&+\frac{H}{H+\mu+r}[1-e^{-(H+\mu+r)(t_2-t_1)}]\frac{\mu}{2\mu+r}(p_1+p_2)\\
=\;&[e^{-(H+\mu+r)(t_2-t_1)}\frac{H\mu^2}{(\mu+r)(H+\mu+r)(2\mu+r)}+\frac{2\mu^2+\mu r+H\mu}{(H+\mu+r)(2\mu+r)}](p_1+p_2).
\end{aligned}$$

Let $A=\frac{H\mu^2}{(\mu+r)(H+\mu+r)(2\mu+r)}$ and $B=\frac{2\mu^2+\mu r+H\mu}{(H+\mu+r)(2\mu+r)}$. At cutoff t_1, the firm is indifferent between disclosing and withholding the solution:

$$\begin{aligned}
[e^{-(H+\mu+r)(t_2-t_1)}A+B](p_1+p_2) &= p_1+\frac{\mu}{2\mu+r}p_2\\
t_2-t_1 &= \frac{\ln\left(\frac{\frac{p_1+\frac{\mu}{2\mu+r}p_2}{p_1+p_2}-B}{A}\right)}{-(H+\mu+r)}.
\end{aligned}$$

Thus, for $p_1H - p_2\mu$ sufficiently large and for r sufficiently small, in every symmetric equilibrium, the difference between t_1 and t_2, or the time length of the withhold region, is constant.

The next step is to show that t_2 is unique as well. Similar to the proof of Theorem 1, we have a breakeven condition by Bayesian updating

$$\begin{aligned}
c &= \tilde{\lambda}(t)H\frac{e^{-H(t-t_1)}\int_t^{\infty}\mu e^{-(\mu+r)(s-t)}ds(p_1+p_2)+\int_0^{t-t_1}He^{-Hs}e^{-(t-t_1-s)\mu}ds\int_t^{\infty}2\mu e^{-2\mu(u-t)}e^{-r(u-t)}du\frac{p_1+p_2}{2}}{e^{-H(t-t_1)}+\int_0^{t-t_1}He^{-Hs}e^{-(t-t_1-s)\mu}ds}\\
c &= \tilde{\lambda}(t)H\frac{e^{-H(t-t_1)}\frac{\mu}{\mu+r}(p_1+p_2)+\int_0^{t-t_1}He^{-Hs}e^{-(t-t_1-s)\mu}ds\frac{\mu}{2\mu+r}(p_1+p_2)}{e^{-H(t-t_1)}+\int_0^{t-t_1}He^{-Hs}e^{-(t-t_1-s)\mu}ds}\\
s.t.\ t_1 &= t-\frac{\ln\left(\frac{\frac{p_1+\frac{\mu}{2\mu+r}p_2}{p_1+p_2}-B}{A}\right)}{-(H+\mu+r)}\text{ if } t\geq\frac{\ln\left(\frac{\frac{p_1+\frac{\mu}{2\mu+r}p_2}{p_1+p_2}-B}{A}\right)}{-(H+\mu+r)};\ t_1=0\text{ otherwise.}
\end{aligned}$$

Next we show that the right-hand side of the above equation is decreasing in t:

$$\begin{aligned}
&\tilde{\lambda}(t)H\frac{e^{-H(t-t_1)}\frac{\mu}{\mu+r}(p_1+p_2)+\int_0^{t-t_1}He^{-Hs}e^{-(t-t_1-s)\mu}ds\frac{\mu}{2\mu+r}(p_1+p_2)}{e^{-H(t-t_1)}+\int_0^{t-t_1}He^{-Hs}e^{-(t-t_1-s)\mu}ds}\\
=\;&\tilde{\lambda}(t)H\{\frac{\mu}{2\mu+r}(p_1+p_2)+\frac{e^{-H(t-t_1)}\left(\frac{\mu}{\mu+r}-\frac{\mu}{2\mu+r}\right)(p_1+p_2)}{e^{-H(t-t_1)}+\int_0^{t-t_1}He^{-Hs}e^{-(t-t_1-s)\mu}ds}\}\\
=\;&\tilde{\lambda}(t)H\{\frac{\mu}{2\mu+r}(p_1+p_2)+\frac{\left(\frac{\mu}{\mu+r}-\frac{\mu}{2\mu+r}\right)(p_1+p_2)}{1+\frac{H}{\mu-H}[1-e^{(H-\mu)(t-t_1)}]}\}
\end{aligned}\qquad(20)$$

For $t \geq \frac{\ln(\frac{\frac{p_1+\frac{\mu}{2\mu+r}p_2}{p_1+p_2}-B}{A})}{-(H+\mu+r)}$, following the proof of Theorem 1 we have a unique characterization of t_1 and t_2.

Part II: prove that a firm will exit at t_2. At $t_2+\Delta t$, staying in the game yields an instantaneous rate of expected payoff

$$
\begin{aligned}
&\tilde{\lambda}(t_2+\Delta t)H\frac{e^{-H(t_2-t_1)}\frac{\mu}{\mu+r}(p_1+p_2)+\int_0^{t_2-t_1}He^{-Hs}e^{-(t_2+\Delta t-t_1-s)\mu}ds\frac{\mu}{2\mu+r}(p_1+p_2)}{e^{-H(t_2-t_1)}+\int_0^{t_2-t_1}He^{-Hs}e^{-(t_2+\Delta t-t_1-s)\mu}ds}\\
&=\frac{\alpha He^{-2Ht_1-H(t_2+\Delta t-t_1)}[e^{-H(t_2-t_1)}\frac{\mu}{\mu+r}(p_1+p_2)+\int_0^{t_2-t_1}He^{-Hs}e^{-(t_2+\Delta t-t_1-s)\mu}ds\frac{\mu}{2\mu+r}(p_1+p_2)]}{\alpha e^{-2Ht_1-H(t_2+\Delta t-t_1)}[e^{-H(t_2-t_1)}+\int_0^{t_2-t_1}He^{-Hs}e^{-(t_2+\Delta t-t_1-s)\mu}ds]+1-\alpha}\\
&=H(p_1+p_2)\frac{\mu}{2\mu+r}\{1+\\
&\frac{e^{-2Ht_1-H(t_2+\Delta t-t_1)}e^{-H(t_2-t_1)}\frac{\frac{\mu}{\mu+r}-\frac{\mu}{2\mu+r}}{\frac{\mu}{2\mu+r}}-\frac{1-\alpha}{\alpha}}{e^{-2Ht_1-H(t_2+\Delta t-t_1)}[e^{-H(t_2-t_1)}+\frac{H}{\mu-H}(e^{-H(t_2-t_1)}-e^{-\mu(t_2-t_1)})e^{-\Delta t\mu}]+\frac{1-\alpha}{\alpha}}\}.
\end{aligned}
\tag{21}
$$

For $\Delta t=0$, (20)=(21), and thus the instantaneous payoff rate is continuous at t_2.

From the breakeven condition of (20), we have

$$
\begin{aligned}
c&=H\frac{e^{-H(t_2-t_1)}\frac{\mu}{\mu+r}(p_1+p_2)+\int_0^{t_2-t_1}He^{-Hs}e^{-(t_2-t_1-s)\mu}ds\frac{\mu}{2\mu+r}(p_1+p_2)}{e^{-H(t_2-t_1)}+\int_0^{t_2-t_1}He^{-Hs}e^{-(t_2-t_1-s)\mu}ds+\frac{1-\alpha}{\alpha}e^{H(t_1+t_2)}}\\
e^{H(t_1+t_2)}&=\frac{\alpha}{1-\alpha}[H\frac{e^{-H(t_2-t_1)}\frac{\mu}{\mu+r}(p_1+p_2)+\frac{H}{\mu-H}(e^{-H(t_2-t_1)}-e^{-\mu(t_2-t_1)})\frac{\mu}{2\mu+r}(p_1+p_2)}{c}\\
&\quad+\frac{\mu}{H-\mu}e^{-H(t_2-t_1)}-\frac{H}{H-\mu}e^{-\mu(t_2-t_1)}]\\
e^{2Ht_1}&=\frac{\alpha}{1-\alpha}[H\frac{e^{-H(t_2-t_1)}\frac{\mu}{\mu+r}(p_1+p_2)+\frac{H}{\mu-H}(e^{-H(t_2-t_1)}-e^{-\mu(t_2-t_1)})\frac{\mu}{2\mu+r}(p_1+p_2)}{c}\\
&\quad+\frac{\mu}{H-\mu}e^{-H(t_2-t_1)}-\frac{H}{H-\mu}e^{-\mu(t_2-t_1)}]e^{H(t_1-t_2)}.
\end{aligned}
$$

If $\alpha<\frac{1}{2}$, we have $e^{2Ht_1}>\frac{\alpha}{1-\alpha}$ and $e^{-2Ht_1}<\frac{1-\alpha}{\alpha}$.

If $\alpha\geq\frac{1}{2}$, we require assumption A3:

$$
c\leq H(p_1+p_2)\frac{\left(\frac{H}{\mu-H}\frac{\mu}{2\mu+r}+\frac{\mu}{\mu+r}\right)e^{-H(t_2-t_1)}-\frac{H}{\mu-H}\frac{\mu}{2\mu+r}e^{-\mu(t_2-t_1)}}{e^{H(t_2-t_1)}-\frac{\mu}{H-\mu}e^{-H(t_2-t_1)}+\frac{H}{H-\mu}e^{-\mu(t_2-t_1)}}
$$

When A3 is satisfied,

$$
\begin{aligned}
e^{2Ht_1}&=\frac{\alpha}{1-\alpha}[H\frac{e^{-H(t_2-t_1)}\frac{\mu}{\mu+r}(p_1+p_2)+\frac{H}{\mu-H}(e^{-H(t_2-t_1)}-e^{-\mu(t_2-t_1)})\frac{\mu}{2\mu+r}(p_1+p_2)}{c}\\
&\quad+\frac{\mu}{H-\mu}e^{-H(t_2-t_1)}-\frac{H}{H-\mu}e^{-\mu(t_2-t_1)}]e^{H(t_1-t_2)}\\
e^{2Ht_1}&\geq\frac{\alpha}{1-\alpha}e^{H(t_2-t_1)}e^{H(t_1-t_2)}\\
e^{2Ht_1}&\geq\frac{\alpha}{1-\alpha}.
\end{aligned}
$$

Following the proof of Theorem 1, we know that

$$
e^{-2Ht_1-H(t_2+\Delta t-t_1)}e^{-H(t_2-t_1)}\frac{\frac{\mu}{\mu+r}-\frac{\mu}{2\mu+r}}{\frac{\mu}{2\mu+r}}-\frac{1-\alpha}{\alpha}\leq 0
$$

is non-positive and decreasing in Δt. Also, the following term

$$
e^{-2Ht_1-H(t_2+\Delta t-t_1)}[e^{-H(t_2-t_1)}+\frac{H}{\mu-H}(e^{-H(t_2-t_1)}-e^{-\mu(t_2-t_1)})e^{-\Delta t\mu}]
$$

is positive and decreasing in Δt. Thus (21) decreases in Δt. We can then conclude that the instantaneous payoff of staying in the game equals c at $\Delta t = 0$ and is smaller than c for all $\Delta t > 0$.

The rest of the proof follows that of Theorem 1.

Part III: prove that a firm will never exit before t_2 .

The above argument has proved that exit is optimal after t_2. For $t \in [0, t_1]$, the instantaneous payoff rate from staying for another dt and disclosing the solution if stage 1 is solved is

$$\tilde{\lambda}(t)[H(p_1 + \frac{\mu}{2\mu + r}p_2) + H\frac{\mu}{2\mu + r}p_2] = \tilde{\lambda}(t)H(p_1 + \frac{2\mu}{2\mu + r}p_2). \tag{22}$$

by (20), c is smaller than $\tilde{\lambda}(t)H\frac{\mu}{\mu+r}(p_1 + p_2)$, and thus (22) must be larger than c.

For $t \in (t_1, t_2]$, the instantaneous payoff rate from continuing research (and adopting a withhold strategy) before solving stage 1, rather than exiting immediately, is

$$\begin{aligned}
&\tilde{\lambda}(t)H\frac{\begin{array}{c}\int_0^{t-t_1} He^{-Hs}e^{-(t-t_1-s)\mu}ds \int_t^{\infty} 2\mu e^{-2\mu(u-t)}e^{-r(u-t)}du\frac{p_1+p_2}{2}\\ +e^{-H(t-t_1)}[e^{-H(t_2-t)}e^{-\mu(t_2-t)}e^{-r(t_2-t)}\int_{t_2}^{\infty}\mu e^{-(\mu+r)(s-t_2)}ds(p_1+p_2)\\ +\int_t^{t_2}\mu e^{-\mu(s-t)}e^{-H(s-t)}e^{-r(s-t)}(p_1+p_2)ds\\ +\int_t^{t_2} He^{-\mu(s-t)}e^{-H(s-t)}e^{-r(s-t)}\int_s^{\infty}2\mu e^{-(2\mu+r)(v-s)}dv\frac{p_1+p_2}{2}ds]\end{array}}{e^{-H(t-t_1)} + \int_0^{t-t_1} He^{-Hs}e^{-(t-t_1-s)\mu}ds}\\
&= H(p_1+p_2)\frac{\begin{array}{c}\frac{H}{\mu-H}[e^{-H(t-t_1)} - e^{-\mu(t-t_1)}]\frac{\mu}{2\mu+r} + e^{-H(t-t_1)}\{e^{-(H+\mu+r)(t_2-t)}\frac{\mu}{\mu+r}\\ +\frac{\mu}{H+\mu+r}[1-e^{-(H+\mu+r)(t_2-t)}] + \frac{H}{H+\mu+r}[1-e^{-(H+\mu+r)(t_2-t)}]\frac{\mu}{2\mu+r}\}\end{array}}{e^{-H(t-t_1)} + \int_0^{t-t_1} He^{-Hs}e^{-(t-t_1-s)\mu}ds + \frac{1-\alpha}{\alpha}e^{2Ht_1+H(t-t_1)}}.
\end{aligned} \tag{23}$$

Omitting the constant term $H(p_1 + p_2)$, the numerator of (23) is

$$\begin{aligned}
&\frac{H}{\mu - H}[e^{-H(t-t_1)} - e^{-\mu(t-t_1)}]\frac{\mu}{2\mu + r} + e^{-H(t-t_1)}\{e^{-(H+\mu+r)(t_2-t)}\frac{\mu}{\mu + r}\\
&+\frac{\mu}{H+\mu+r}[1 - e^{-(H+\mu+r)(t_2-t)}] + \frac{H}{H+\mu+r}[1 - e^{-(H+\mu+r)(t_2-t)}]\frac{\mu}{2\mu + r}\}\\
&= e^{-H(t-t_1)}[\frac{H}{\mu - H}\frac{\mu}{2\mu + r} - \frac{H}{\mu - H}\frac{\mu}{2\mu + r}e^{(H-\mu)(t-t_1)}\\
&+\left(\frac{\mu}{H+\mu+r} + \frac{H}{H+\mu+r}\frac{\mu}{2\mu+r}\right)\\
&+e^{-(H+\mu+r)(t_2-t)}\left(\frac{\mu}{\mu+r} - \frac{\mu}{H+\mu+r} - \frac{H}{H+\mu+r}\frac{\mu}{2\mu+r}\right)],
\end{aligned}$$

whose derivative with respect to t is

$$\begin{aligned}
&-He^{-H(t-t_1)}(\frac{H}{\mu-H}\frac{\mu}{2\mu+r} + \frac{\mu}{H+\mu+r}\\
&+\frac{H}{H+\mu+r}\frac{\mu}{2\mu+r}) + \mu e^{-\mu(t-t_1)}\frac{H}{\mu-H}\frac{\mu}{2\mu+r}\\
&+(\mu+r)e^{-H(t_2-t_1)-(\mu+r)(t_2-t)}(\frac{\mu}{\mu+r} - \frac{\mu}{H+\mu+r} - \frac{H}{H+\mu+r}\frac{\mu}{2\mu+r})\\
&= -He^{-H(t-t_1)}[(\frac{H}{\mu-H}+1)\frac{\mu}{2\mu+r} + \frac{\mu}{H+\mu+r}\\
&+(\frac{H}{H+\mu+r}-1)\frac{\mu}{2\mu+r}] + \mu e^{-\mu(t-t_1)}\frac{H}{\mu-H}\frac{\mu}{2\mu+r}\\
&+e^{-H(t_2-t_1)-(\mu+r)(t_2-t)}(\mu - \frac{(\mu+r)\mu}{H+\mu+r} - \frac{H(\mu+r)}{H+\mu+r}\frac{\mu}{2\mu+r})\\
&= -[e^{-H(t-t_1)} - e^{-\mu(t-t_1)}](\frac{H\mu}{\mu-H}\frac{\mu}{2\mu+r})\\
&[e^{-H(t_2-t_1)-(\mu+r)(t_2-t)} - e^{-H(t-t_1)}](\frac{H\mu}{H+\mu+r} + \frac{-H(\mu+r)}{H+\mu+r}\frac{\mu}{2\mu+r}).
\end{aligned} \tag{24}$$

The second term of (24) is smaller than 0 since $-H(t - t_1) > -H(t_2 - t_1) - (\mu + r)(t_2 - t)$. Also, the first term is smaller than 0. Hence the numerator of (23) is decreasing in t.

By the same argument as in the proof of Theorem 1, when A3 is satisfied, the denominator of (23) is increasing in t. In conclusion, (23) is decreasing in t for $t \in [t_1, t_2]$. Notice that (23) $= c$ at $t = t_2$. Thus the value of staying is always larger than the cost, and the firm will not exit.

Part IV: prove that withholding the solution after solving stage 1 is optimal in the withhold region.

At time $t \in [t_1, t_2)$, the expected payoff from disclosure is

$$\frac{(p_1 + \frac{\mu}{2\mu+r}p_2)e^{-H(t-t_1)} + \left(\frac{p_1}{2} + \frac{\mu}{2\mu+r}p_2\right)\int_{t_1}^{t} He^{-H(s-t_1)}e^{-\mu(t-s)}ds}{e^{-H(t-t_1)} + \int_{t_1}^{t} He^{-H(s-t_1)}e^{-\mu(t-s)}ds}.$$

The expected payoff from withholding the solution is

$$\frac{\begin{array}{c}\int_{t_1}^{t} He^{-H(s-t_1)}e^{-\mu(t-s)}ds\frac{\mu}{2\mu+r}(p_1+p_2) + e^{-H(t-t_1)}[e^{-(H+\mu+r)(t_2-t)}\frac{\mu}{\mu+r}(p_1+p_2)\\ +\int_{t}^{t_2}\mu e^{-(H+\mu+r)(s-t_1)}ds(p_1+p_2) + \int_{t_1}^{t_2} He^{-(H+\mu+r)(s-t_1)}ds\frac{\mu}{2\mu+r}(p_1+p_2)]\end{array}}{e^{-H(t-t_1)} + \int_{t_1}^{t} He^{-H(s-t_1)}e^{-\mu(t-s)}ds},$$

and the difference between withholding and disclosing the solution is

$$\frac{\begin{array}{c}\int_{t_1}^{t} He^{-H(s-t_1)}e^{-\mu(t-s)}ds\frac{\mu}{2\mu+r}(p_1+p_2) + e^{-H(t-t_1)}[e^{-(H+\mu+r)(t_2-t)}\frac{\mu}{\mu+r}\\ +\int_{t}^{t_2}\mu e^{-(H+\mu+r)(s-t_1)}ds + \int_{t_1}^{t_2} He^{-(H+\mu+r)(s-t_1)}ds\frac{\mu}{2\mu+r}](p_1+p_2)\\ -(p_1 + \frac{\mu}{2\mu+r}p_2)e^{-H(t-t_1)} - \left(\frac{p_1}{2} + \frac{\mu}{2\mu+r}p_2\right)\int_{t_1}^{t} He^{-H(s-t_1)}e^{-\mu(t-s)}ds\end{array}}{e^{-H(t-t_1)} + \int_{t_1}^{t} He^{-H(s-t_1)}e^{-\mu(t-s)}ds}$$

$$= \frac{\begin{array}{c}\int_{t_1}^{t} He^{-H(s-t_1)}e^{-\mu(t-s)}ds\frac{-\frac{1}{2}r}{2\mu+r}p_1\\ +e^{-H(t-t_1)}[e^{-(H+\mu+r)(t_2-t)}\frac{H\mu^2}{(\mu+r)(H+\mu+r)(2\mu+r)}(p_1+p_2)\\ +\frac{2\mu^2+\mu r+H\mu}{(H+\mu+r)(2\mu+r)}(p_1+p_2) - (p_1 + \frac{\mu}{2\mu+r}p_2)]\end{array}}{e^{-H(t-t_1)} + \int_{t_1}^{t} He^{-H(s-t_1)}e^{-\mu(t-s)}ds}$$

$$= \frac{\begin{array}{c}\frac{\frac{1}{2}r}{2\mu+r}\frac{p_1H}{H-\mu}[1-e^{(H-\mu)(t-t_1)}] + [e^{-(H+\mu+r)(t_2-t)}\frac{H\mu^2}{(\mu+r)(H+\mu+r)(2\mu+r)}(p_1+p_2)\\ +\frac{-r^2-H\mu-2\mu r-rH}{(H+\mu+r)(2\mu+r)}p_1 + \frac{\mu^2}{(H+\mu+r)(2\mu+r)}p_2)]\end{array}}{1-\frac{H}{H-\mu}[1-e^{(H-\mu)(t-t_1)}]}. \tag{25}$$

The numerator is increasing in t. The denominator is always positive. Noticably, by our definition of equilibrium, (25) is zero at $t = t_1$. Then as t increases from t_1, the numerator is always positive. Thus (25) > 0 for $t \in [t_1, t_2)$. Hence, withholding the solution is confirmed to be superior to disclosure everywhere in the withhold region.

Part V: prove that disclosure is optimal after solving stage 1 in the disclose region.

We discuss two types of deviation: type-1 deviation where the firm will withhold the solution for some period and then disclose the solution, and type-2 deviation where the firm will withhold the solution until it solves stage 2. We verify that both types of deviation are inferior to immediate disclosure. From our Part I, in the disclose region, i.e., $[0, t_1)$, the payoff from disclosure is $p_1 + \frac{\mu}{2\mu+r}p_2$.

Suppose that the firm uses type-1 deviation. By the proof of Lemma 4, the payoff is always less than $p_1 + \frac{\mu}{2\mu+r}p_2$, and thus any type-1 deviation is not profitable.

Suppose that the firm uses type-2 deviation, and let U denote the expected payoff at time t_1; then the expected payoff at time t is

$$\begin{aligned}&e^{-(H+\mu+r)(t_1-t)}U + \int_{t}^{t_1}\mu e^{-(H+\mu+r)(s-t)}ds(p_1+p_2)\\ &+\int_{t}^{t_1} He^{-(H+\mu+r)(s-t)}ds\left(\frac{1}{2}p_1 + \frac{\mu}{2\mu+r}p_2\right).\end{aligned} \tag{26}$$

The profit generated by the deviation is (26) minus the original payoff $p_1 + \frac{\mu}{2\mu+r}p_2$. However, the firm should be indifferent between disclosing and withholding the solution at t_1; thus $U = p_1 + \frac{\mu}{2\mu+r}p_2$. By the proof of Lemma 3 $'$, no type-2 deviation is profitable. Hence, disclosure is confirmed to be superior to withholding the solution everywhere in the disclose region. This completes the proof.

Appendix F. Proof of Proposition 2

Suppose that there exists a symmetric equilibrium with a disclose region of positive length. By Lemmas 1–3, we know that this equilibrium must exhibit a disclose-withhold-exit pattern. Let t_1 and t_2 denote the beginning of the withhold region and the exit point respectively.

First, we prove that in this equilibrium, if a firm will exit at t_2, it will never disclose the solution after t_2.

At $t > t_2$, disclosure yields payoff $p_1 + \frac{\mu}{2\mu+r}p_2$, while withholding ever afterwards yields payoff

$$\frac{e^{-H(t_2-t_1)}\frac{\mu}{\mu+r}(p_1+p_2) + \int_{t_1}^{t_2} He^{-H(s-t_1)}e^{-\mu(t-s)}ds\frac{\mu}{2\mu+r}(p_1+p_2)}{e^{-H(t_2-t_1)} + \int_{t_1}^{t_2} He^{-H(s-t_1)}e^{-\mu(t-s)}ds}.$$

The difference between withholding and disclosing the solution is

$$\frac{\begin{array}{c} e^{-H(t_2-t_1)}\frac{\mu}{\mu+r}(p_1+p_2)+\int_{t_1}^{t_2}He^{-H(s-t_1)}e^{-\mu(t-s)}ds\frac{\mu}{2\mu+r}(p_1+p_2) \\ -e^{-H(t_2-t_1)}(p_1+\frac{\mu}{2\mu+r}p_2)-\int_{t_1}^{t_2}He^{-H(s-t_1)}e^{-\mu(t-s)}ds(p_1+\frac{\mu}{2\mu+r}p_2)\end{array}}{e^{-H(t_2-t_1)}+\int_{t_1}^{t_2}He^{-H(s-t_1)}e^{-\mu(t-s)}ds}.$$

Lemma 3 implies that at $t=t_2$, withholding is preferred to disclosure, i.e. the above difference is non-negative. Moreover, the denominator of the above expression is positive, and the derivative of the numerator with respect to t is

$$\begin{aligned} & e^{-H(t_2-t_1)}\frac{H\mu}{H-\mu}[e^{\mu(t_2-t)}-e^{H(t_2-t_1)+\mu(t_1-t)}]\frac{-\mu-r}{2\mu+r}(p_1+p_2) \\ =\ & e^{-H(t_2-t_1)}\frac{H\mu}{H-\mu}e^{\mu(t_2-t)}[1-e^{(H-\mu)(t_2-t_1)}]\frac{-\mu-r}{2\mu+r}(p_1+p_2) \\ \geq\ & 0. \end{aligned}$$

Thus withholding the solution is always preferred to disclosure.

Next, note that in equilibrium withholding the solution must dominate disclosure in the withhold region. For every $t\in[t_1,t_2]$, disclosure yields payoff $p_1+\frac{\mu}{2\mu+r}p_2$, and withholding ever afterwards yields payoff

$$\frac{\begin{array}{c}\int_{t_1}^{t}He^{-H(s-t_1)}e^{-\mu(t-s)}ds\frac{\mu}{2\mu+r}(p_1+p_2)+e^{-H(t-t_1)}[e^{-(H+\mu+r)(t_2-t)}\frac{\mu}{\mu+r}(p_1+p_2) \\ +\int_{t}^{t_2}\mu e^{-(H+\mu+r)(s-t_1)}ds(p_1+p_2)+\int_{t_1}^{t_2}He^{-(H+\mu+r)(s-t_1)}ds\frac{\mu}{2\mu+r}(p_1+p_2)]\end{array}}{e^{-H(t-t_1)}+\int_{t_1}^{t}He^{-H(s-t_1)}e^{-\mu(t-s)}ds}.$$

The difference between withholding and disclosing the solution is

$$\begin{aligned} & \frac{\begin{array}{c}\int_{t_1}^{t}He^{-H(s-t_1)}e^{-\mu(t-s)}ds\frac{\mu}{2\mu+r}(p_1+p_2)+e^{-H(t-t_1)}[e^{-(H+\mu+r)(t_2-t)}\frac{\mu}{\mu+r}(p_1+p_2) \\ +\int_{t}^{t_2}\mu e^{-(H+\mu+r)(s-t_1)}ds(p_1+p_2)+\int_{t_1}^{t_2}He^{-(H+\mu+r)(s-t_1)}ds\frac{\mu}{2\mu+r}(p_1+p_2)] \\ -e^{-H(t-t_1)}(p_1+\frac{\mu}{2\mu+r}p_2)-\int_{t_1}^{t}He^{-H(s-t_1)}e^{-\mu(t-s)}ds(p_1+\frac{\mu}{2\mu+r}p_2)\end{array}}{e^{-H(t-t_1)}+\int_{t_1}^{t}He^{-H(s-t_1)}e^{-\mu(t-s)}ds} \\ =\ & \frac{\begin{array}{c}-\frac{H}{H-\mu}[1-e^{(H-\mu)(t-t_1)}]\frac{\mu}{2\mu+r}(p_1+p_2)+[e^{-(H+\mu+r)(t_2-t)}\frac{H\mu^2}{(\mu+r)(H+\mu+r)(2\mu+r)}(p_1+p_2) \\ +\frac{2\mu^2+\mu r+H\mu}{(H+\mu+r)(2\mu+r)}(p_1+p_2)-(p_1+\frac{\mu}{2\mu+r}p_2)]-\{-\frac{H}{H-\mu}[1-e^{(H-\mu)(t-t_1)}](p_1+\frac{\mu}{2\mu+r}p_2)\}\end{array}}{1-\frac{H}{H-\mu}[1-e^{(H-\mu)(t-t_1)}]} \\ =\ & \frac{\begin{array}{c}\frac{H}{H-\mu}[1-e^{(H-\mu)(t-t_1)}]\frac{\mu+r}{2\mu+r}p_1+[e^{-(H+\mu+r)(t_2-t)}\frac{H\mu^2}{(\mu+r)(H+\mu+r)(2\mu+r)}(p_1+p_2) \\ +\frac{-r^2-H\mu-2\mu r-rH}{(H+\mu+r)(2\mu+r)}p_1+\frac{\mu^2}{(H+\mu+r)(2\mu+r)}p_2)]\end{array}}{1-\frac{H}{H-\mu}[1-e^{(H-\mu)(t-t_1)}]} \end{aligned} \tag{27}$$

The denominator is always positive. Take derivative of the numerator with respect to t, and we have the expression

$$-He^{(H-\mu)(t-t_1)}\frac{\mu+r}{2\mu+r}p_1+e^{-(H+\mu+r)(t_2-t)}\frac{H\mu^2}{(\mu+r)(2\mu+r)}(p_1+p_2).$$

The above expression must be non-negative at $t=t_1$. Thus we require

$$\begin{aligned} & e^{-(H+\mu+r)(t_2-t)}\frac{H\mu^2}{(\mu+r)(2\mu+r)}(p_1+p_2)\geq He^{(H-\mu)(t-t_1)}\frac{\mu+r}{2\mu+r}p_1 \\ \Leftrightarrow\ & e^{-(H+\mu+r)(t_2-t)}\left(\frac{\mu}{\mu+r}\right)^2(p_1+p_2)\geq e^{(H-\mu)(t-t_1)}p_1. \end{aligned}$$

at $t=t_1$, which implies that

$$\begin{aligned} & e^{-(H+\mu+r)(t_2-t_1)}(\frac{\mu}{\mu+r})^2(p_1+p_2)\geq p_1 \\ \Leftrightarrow\ & \frac{\frac{p_1+\frac{\mu}{2\mu+r}p_2}{p_1+p_2}-B}{A}(\frac{\mu}{\mu+r})^2(p_1+p_2)\geq p_1 \\ \Leftrightarrow\ & \frac{\frac{p_1+\frac{\mu}{2\mu+r}p_2}{p_1+p_2}-\frac{2\mu^2+\mu r+H\mu}{(H+\mu+r)(2\mu+r)}}{H\mu^2}(\mu+r)(H+\mu+r)(2\mu+r)\geq\frac{p_1}{p_1+p_2}(\frac{\mu+r}{\mu})^2 \\ \Leftrightarrow\ & [p_1+\frac{\mu}{2\mu+r}p_2-\frac{2\mu^2+\mu r+H\mu}{(H+\mu+r)(2\mu+r)}(p_1+p_2)](H+\mu+r)(2\mu+r) \\ & \geq H(\mu+r)p_1 \end{aligned}$$

$$\begin{aligned}&\Leftrightarrow (2H\mu+2\mu^2+2\mu r+Hr+\mu r+r^2-2\mu^2-\mu r-H\mu-H\mu-Hr)p_1\\&\quad\geq(2\mu^2+\mu r+H\mu-H\mu-\mu^2-r\mu)p_2\\&\Leftrightarrow (2\mu r+r^2)p_1\geq\mu^2 p_2.\end{aligned}$$

However, $(2\mu r+r^2)p_1>\mu^2 p_2$ violates assumption A2. Hence, disclosure will never occur in any symmetric equilibrium.

Appendix G. Proof of Proposition 3

As in the proof of Theorem 1, we proceed by five steps.

Part I: characterize t_1 and t_2.

Lemma 4. *Fix t_1. t_2-t_1 is unique for firm n to be indifferent between disclosing and withholding the solution at t_1.*

Proof. Let $V_0(t_1|t_2)$ denote each firm's expected payoff at t_1 from withholding the solution. We have

$$V_0(t_1|t_2)=P_0(t_1|t_2)(p_1+p_2)+\int_{t_1}^{t_2}V_1(t^0|t_2)dF_1(t^0|t_2).$$

In this equation, $P_0(t_1|t_2)$ is the probability of event E_0: firm n solves stage 2 before any of its competitors solves stage 1; $V_1(t^0|t_2)$ is firm n's continuation payoff at t^0, when one of its competitors $1,\cdots,n-1$, without loss of generality assumed to be firm 1, first solves stage 1 at $t^0\in[t_1,t_2]$; $F_1(t^0|t_2)$ is the (unconditional) distribution of t^0, i.e. $F_1(t^0=t_2|t_2)=1-P_0(t_1|t_2)$. Note that $F_1(t^0|t_2)=F_1(t^0|t_2')$ whenever $t^0\in[t_1,t_2]$ and $t_2'\geq t_2$, and that $\frac{dP_0(t_1|t_2)}{dt_2}=f(t^0=t_2|t_2)$ where f denotes the probability density function of F.

We know that $V_1(t^0|t_2)<(p_1+p_2)$ for every t^0. Also,

$$\begin{aligned}P_0(t_1|t_2)&=e^{-((n-1)H+\mu)(t_2-t_1)}+\int_0^{t_2-t_1}e^{-(n-1)Ht}\mu e^{-\mu t}dt\\&=e^{-((n-1)H+\mu)(t_2-t_1)}+\frac{\mu}{(n-1)H+\mu}(1-e^{-((n-1)H+\mu)(t_2-t_1)})\\&=\frac{\mu}{(n-1)H+\mu}+\frac{(n-1)H}{(n-1)H+\mu}e^{-((n-1)H+\mu)(t_2-t_1)},\end{aligned}$$

which is decreasing in t_2. Hence, for $V_0(t_1|t_2)$ to decrease in t_2, it suffices to show that $V_1(t^0|t_2)$ decreases in t_2 for every t^0. Likewise, we can decompose $V_1(t^0|t_2)$:

$$V_1(t^0|t_2)=P_1(t^0|t_2)\frac{p_1+p_2}{2}+\int_{t^0}^{t_2}V_2(t^1|t_2)dF_2(t^1|t_2).$$

$P_1(t^0|t_2)$ is the probability of event E_1: firm n or firm 1 solves stage 2 before any of their competitors solves stage 1; $V_2(t^1|t_2)$ is firm n's continuation payoff at t^1, when one of the competitors $2,\cdot,n-1$, without loss of generality assumed to be firm 2, first solves stage 1 at $t^1\in[t_1,t_2]$; $F_2(t^1|t_2)$ is the distribution of t^1.

We know that $V_2(t^1|t_2)<\frac{p_1+p_2}{2}$ for every t^1. Also,

$$\begin{aligned}P_1(t^0|t_2)&=e^{-((n-2)H+2\mu)(t_2-t^0)}+\int_0^{t_2-t^0}e^{-(n-2)Ht}2\mu e^{-2\mu t}dt\\&=e^{-((n-1)H+2\mu)(t_2-t^0)}+\frac{2\mu}{(n-2)H+2\mu}(1-e^{-((n-2)H+2\mu)(t_2-t^0)})\\&=\frac{2\mu}{(n-2)H+2\mu}+\frac{(n-2)H}{(n-2)H+2\mu}e^{-((n-2)H+2\mu)(t_2-t^0)},\end{aligned}$$

which is decreasing in t_2. Hence, for $V_1(t^0|t_2)$ to decrease in t_2, it suffices to show that $V_2(t^1|t_2)$ decreases in t_2 for every t^1.

The argument then unravels forward. At the final step, we have

$$V_{n-2}(t^{n-3}|t_2)=P_{n-2}(t^{n-3}|t_2)\frac{p_1+p_2}{n-1}+\int_{t^{n-3}}^{t_2}V_{n-1}(t^{n-2}|t_2)dF_{n-1}(t^{n-2}|t_2).$$

$V_{n-1}(t^{n-2}|t_2)$, which is firm n's continuation payoff at t^{n-2} when the remaining firm $n-1$ solves stage 1 at $t^{n-2}\in[t^{n-3},t_2]$, is a constant $\frac{p_1+p_2}{n}<\frac{p_1+p_2}{n-1}$. Hence,

$$V_{n-2}(t^{n-3}|t_2)=P_{n-2}(t^{n-3}|t_2)\frac{p_1+p_2}{n-1}+(1-P_{n-2}(t^{n-3}|t_2))\frac{p_1+p_2}{n}.$$

Also,

$$P_{n-2}(t^{n-3}|t_2)=e^{-(H+(n-1)\mu)(t_2-t^{n-3})}+\int_0^{t_2-t^{n-3}}e^{-Ht}(n-1)\mu e^{-(n-1)\mu t}dt$$

$$= e^{-(H+(n-1)\mu)(t_2-t^{n-3})} + \frac{(n-1)\mu}{H+(n-1)\mu}(1-e^{-(H+(n-1)\mu)(t_2-t^{n-3})})$$
$$= \frac{(n-1)\mu}{H+(n-1)\mu} + \frac{H}{H+(n-1)\mu}e^{-(H+(n-1)\mu)(t_2-t^{n-3})},$$

which is decreasing in t_2. Hence $V_{n-2}(t^{n-3}|t_2)$ decreases in t_2, and by our inductive argument $V_0(t_1|t_2)$ decreases in t_2.

Note that for firm n to be indifferent, a necessary and sufficient condition is $p_1 + \frac{p_2}{n} = V_0(t_1|t_2)$. Since $V_0(t_1|t_2)$ decreases in t_2, and in addition $\lim_{t_2\to t_1^+} V_0(t_1|t_2) = p_1 + p_2 > p_1 + \frac{p_2}{n}$ and $\lim_{t_2\to\infty} V_0(t_1|t_2) = \frac{p_1+p_2}{n} < p_1 + \frac{p_2}{n}$, the required $t_2 - t_1$ must be unique. □

From here on we fix $t_2 - t_1 = \nabla$, and we prove that t_2 is unique as well. In absence of any breakthrough or disclosure, each firm's belief at t_2 is

$$\tilde{\lambda}(t_2) = \frac{\alpha e^{-nH(t_2-\nabla)-H\nabla}(e^{-H\nabla} + \int_0^\nabla He^{-Hs}e^{-(\nabla-s)\mu}ds)^{n-1}}{\alpha e^{-nH(t_2-\nabla)-H\nabla}(e^{-H\nabla} + \int_0^\nabla He^{-Hs}e^{-(\nabla-s)\mu}ds)^{n-1} + 1-\alpha}$$

Fix ∇, as t_2 and t_1 simultaneously increases, $\tilde{\lambda}(t_2)$ decreases.

At t_2, the instantaneous payoff of research is

$$\tilde{\lambda}(t_2)H\frac{\sum_{m=0}^{n-1}C_m^{n-1}(e^{-H\nabla})^{n-m-1}(\int_0^\nabla He^{-Hs}e^{-(\nabla-s)\mu}ds)^m(\frac{p_1+p_2}{m+1})}{(e^{-H\nabla} + \int_0^\nabla He^{-Hs}e^{-(\nabla-s)\mu}ds)^{n-1}}$$

where $C_m^{n-1} = \frac{(n-1)!}{m!(n-1-m)!}$ is the binomial coefficient.

As $\tilde{\lambda}(t_2)$ decreases in t_2, the instantaneous payoff decreases in t_2. Note that in equilibrium, the instantaneous payoff should be equal to c at t_2, thus we can find a unique value of t_2 such that the instantaneous payoff above equals to c. With a little abuse of notation, from here on we use t_2 to denote this unique solution.

Part II: prove that a firm will exit at t_2. In the case of $n = 2$, we require assumption A3, under which each firm's instantaneous payoff rate at $t_2 + \Delta t$ is decreasing in Δt, to guarantee the existence of an equilibrium. For an arbitrary n, as the closed-form expressions of t_1 and t_2 are too complex, we hereby introduce an alternative assumption that bears the same spirit as A3. The "m" below stands for multiple firms.

A3(m) Let $A = e^{-H(t_2-t_1)}$ and $B = \int_0^{t_2-t_1} He^{-Hs-\mu(t_2-t_1-s)}ds = \frac{H}{H-\mu}(e^{-\mu(t_2-t_1)} - e^{-H(t_2-t_1)})$. The condition below is satisfied:

$$\frac{A^2((A+B)^{n-1} - A^{n-1}))}{Be^{nHt_1}} \le \frac{1-\alpha}{\alpha}.$$

Lemma 5. *Assume A3(m). If any firm stays in the competition for Δt after t_2, its instantaneous payoff rate at $t_2 + \Delta t$ is decreasing in Δt.*

Proof. Let $B' = \int_0^{t_2-t_1} He^{-Hs-\mu(t_2-t_1+\Delta t-s)}ds < B$. The above instantaneous payoff rate is equal to

$$\tilde{\lambda}(t_2+\Delta t)H\frac{\sum_{m=0}^{n-1}C_{n-1}^m A^{n-m-1}B'^m\frac{p_1+p_2}{m+1}}{(A+B')^{n-1}},$$

where

$$\tilde{\lambda}(t_2+\Delta t) = \frac{\alpha(A+B')^{n-1}Ae^{-H(nt_1+\Delta t)}}{\alpha(A+B')^{n-1}Ae^{-H(nt_1+\Delta t)} + 1-\alpha}.$$

Hence the instantaneous payoff rate can be written as

$$H(p_1+p_2)\frac{Ae^{-H(nt_1+\Delta t)}\sum_{m=0}^{n-1}\frac{C_{n-1}^m A^{n-m-1}B'^m}{m+1}}{Ae^{-H(nt_1+\Delta t)}(A+B')^{n-1} + \frac{1-\alpha}{\alpha}}.$$

Note that

$$nB'\sum_{m=0}^{n-1}\frac{C_{n-1}^m A^{n-m-1}B'^m}{m+1} = \sum_{m=0}^{n-1}C_n^{m+1}A^{n-m-1}B'^{m+1} = (A+B')^n - A^n,$$

and thus we can write the instantaneous payoff rate as

$$H(p_1+p_2)\frac{Ae^{-H(nt_1+\Delta t)}\frac{(A+B')^n-A^n}{nB'}}{Ae^{-H(nt_1+\Delta t)}(A+B')^{n-1} + \frac{1-\alpha}{\alpha}}$$
$$= \frac{H(p_1+p_2)}{n}(1 + \frac{Ae^{-H(nt_1+\Delta t)}(\frac{(A+B')^n-A^n}{B'} - (A+B')^{n-1}) - \frac{1-\alpha}{\alpha}}{Ae^{-H(nt_1+\Delta t)}(A+B')^{n-1} + \frac{1-\alpha}{\alpha}})$$

$$=\frac{H(p_1+p_2)}{n}\left(1+\frac{\frac{A}{B'}e^{-H(nt_1+\Delta t)}(A(A+B')^{n-1}-A^n)-\frac{1-\alpha}{\alpha}}{Ae^{-H(nt_1+\Delta t)}(A+B')^{n-1}+\frac{1-\alpha}{\alpha}}\right)$$
$$=\frac{H(p_1+p_2)}{n}\left(1+\frac{\frac{A^2}{B'}e^{-H(nt_1+\Delta t)}((A+B')^{n-1}-A^{n-1})-\frac{1-\alpha}{\alpha}}{Ae^{-H(nt_1+\Delta t)}(A+B')^{n-1}+\frac{1-\alpha}{\alpha}}\right)$$

In the second fraction, the denominator is positive and decreasing in Δt. Also, $\frac{(A+B')^{n-1}-A^{n-1}}{B'}$ is decreasing in B', which is decreasing in Δt, which further implies that the numerator is decreasing in Δt as well. Therefore it suffices to prove that the numerator is always non-positive. Note that A3(m) is equivalent to the numerator being non-positive when $\Delta t = 0$. Hence, the numerator is non-positive for all $\Delta t \geq 0$. This completes the proof. □

We would like to highlight here two natural requirements on the model parameters which lead to A3(m). First, t_1 is bounded below by 0, A^2 is bounded above by 1, and

$$\frac{(A+B)^{n-1}-A^{n-1}}{B}=\sum_{m=1}^{n-1}C_{n-1}^{m}B^{m-1}A^{n-1-m}=\sum_{m=1}^{n-1}\frac{n-1}{m}C_{n-2}^{m-1}B^{m-1}A^{n-2-(m-1)}$$
$$=\sum_{m=0}^{n-2}\frac{n-1}{m+1}C_{n-2}^{m}B^{m}A^{n-2-m}\leq(n-1)\sum_{m=0}^{n-2}C_{n-2}^{m}B^{m}A^{n-2-m}=(n-1)(A+B)^{n-2}\leq n-1.$$

Therefore, A3(m) is satisfied when $n-1\leq\frac{1-\alpha}{\alpha}$, i.e. $\alpha\leq\frac{1}{n}$. Second, both A and B are constants as long as $t_1>0$, so A3(m) is also satisfied for any given α when c, the research cost for stage 1, is sufficiently low. These two requirements also correspond to the two conditions in A3. When $n=2$, the second requirement is even weaker than the respective condition in A3.

Part III: prove that a firm will never exit before t_2**.** For $t\in[0,t_1]$ the proof is similar to the proof of Theorem 1 and is trivial. For $t\in(t_1,t_2]$ the instantaneous payoff rate from continuing research (withhold if a breakthrough arrives), rather than exiting immediately, is

< PL2Xconstructcontent − type = "equation" >< p > PLX − 0 − PLX < /ce : para >< /PL2Xconstruct > (28)

where $V(t)$ is the firm's continuation payoff conditional on stage 1 being solvable ($\lambda=H$).

We rewrite the above condition in the same spirit of the proof of Lemma 1 by, without loss of generality, focusing on firm n:

$$V(t)\left[\sum_{m=0}^{n-1}C_{n-1}^{m}\left(\int_{t_1}^{t}He^{-H(s-t_1)}e^{-\mu(t-s)}ds\right)^m\left(e^{-H(t-t_1)}\right)^{n-m-1}+\frac{1-\alpha}{\alpha}e^{nHt_1+H(t-t_1)}\right]$$
$$=P_0(t)(p_1+p_2)+\int_{t_1}^{t_2}V_1(t^0)dF_1(t^0).$$

In this equation, $P_0(t)$ is the probability of event E_0: firm n solves stage 2 before any of its competitors solves stage 1; $V_1(t^0)$ is firm n's continuation payoff at t^0, when one of its competitors $1,\cdots,n-1$, without loss of generality assumed to be firm 1, first solves stage 1 at $t^0\in[t_1,t_2]$; $F_1(t^0)$ is the (unconditional) distribution of t^0.

We know that $V_1(t^0)<(p_1+p_2)$ for every t^0. Also,

$$P_0(t)=e^{-(n-1)H(t_2-t_1)}e^{-\mu(t_2-t)}+\int_t^{t_2}e^{-(n-1)H(s-t_1)}\mu e^{-\mu(s-t)}ds$$
$$=e^{-(n-1)H(t_2-t_1)}e^{-\mu(t_2-t)}-\frac{\mu}{(n-1)H+\mu}\left[e^{-(n-1)H(t_2-t_1)}e^{-\mu(t_2-t)}-e^{-(n-1)H(t-t_1)}\right]$$
$$=\frac{(n-1)H}{(n-1)H+\mu}e^{-(n-1)H(t_2-t_1)}e^{-\mu(t_2-t)}+\frac{\mu}{(n-1)H+\mu}e^{-(n-1)H(t-t_1)}$$
$$P_0'(t)=-\frac{(n-1)H\mu}{(n-1)H+\mu}\left[e^{-(n-1)H(t-t_1)}-e^{-(n-1)H(t_2-t_1)}e^{-\mu(t_2-t)}\right]<0$$

$P_0(t)$ is decreasing in t. Hence, for $V_0(t)$ to decrease in t, it suffices to show that $V_1(t^0)$ decreases in t for every t^0. Likewise, we decompose $V_1(t^0)$

$$V_1(t^0)=P_1(t^0)\frac{p_1+p_2}{2}+\int_{t^0}^{t_2}V_2(t^1)dF_2(t^1)$$

$P_1(t^0)$ is the probability of event E_1: firm n or firm 1 solves stage 2 before any of their competitors solves stage 1; $V_2(t^1)$ is firm n's continuation payoff at t^1, when one of the competitors $2,\cdot,n-1$, without loss of generality assumed to be firm 2, first solves stage 1 at $t^1\in[t_1,t_2]$; $F_2(t^1)$ is the distribution of t^1.

We know that $V_2(t^1)<\frac{p_1+p_2}{2}$ for every t^1. Also, for $t^0\leq t$

$$P_1(t^0)=e^{-(n-2)H(t_2-t^0)}e^{-2\mu(t_2-t)}+\int_t^{t_2}e^{-(n-2)H(s-t^0)}2\mu e^{-2\mu(s-t)}ds$$

$$= e^{-(n-2)H(t_2-t^0)}e^{-2\mu(t_2-t)} - \frac{2\mu}{(n-2)H+2\mu}[e^{-(n-2)H(t_2-t^0)}e^{-2\mu(t_2-t)} - e^{-(n-2)H(t-t^0)}]$$
$$= \frac{(n-2)H}{(n-2)H+2\mu}e^{-(n-2)H(t_2-t^0)}e^{-2\mu(t_2-t)} + \frac{2\mu}{(n-2)H+2\mu}e^{-(n-2)H(t-t^0)}$$
$$P_1'(t^0) = -\frac{2(n-2)H\mu}{(n-2)H+2\mu}[e^{-(n-2)H(t-t^0)} - e^{-(n-2)H(t_2-t^0)}e^{-2\mu(t_2-t)}] < 0$$

$P_1'(t^0)$ is decreasing in t; for $t^0 > t$

$$P_1(t^0) = e^{-(n-2)H(t_2-t^0)}e^{-2\mu(t_2-t^0)} + \int_{t^0}^{t_2} e^{-(n-2)H(s-t^0)}2\mu e^{-2\mu(s-t_0)}ds$$

is not changing in t. Hence, for $V_1(t^0)$ to decrease in t, it suffices to show that $V_2(t^1)$ decreases in t for every t^1. The argument then unravels forward. At the final step, we have

$$V_{n-2}(t^{n-3}) = P_{n-2}(t^{n-3})\frac{p_1+p_2}{n-1} + (1-P_{n-2}(t^{n-3}))\frac{p_1+p_2}{n}.$$

For $t^{n-3} \le t$,

$$P_{n-2}(t^{n-3}) = e^{-H(t_2-t^{n-3})}e^{-(n-1)\mu(t_2-t)} + \int_t^{t_2} e^{-H(s-t^{n-3})}(n-1)\mu e^{-(n-1)\mu(s-t)}ds$$
$$= e^{-H(t_2-t^{n-3})}e^{-(n-1)\mu(t_2-t)} - \frac{(n-1)\mu}{H+(n-1)\mu}[e^{-H(t_2-t^{n-3})}e^{-(n-1)\mu(t_2-t)} - e^{-H(t-t^{n-3})}]$$
$$= \frac{H}{H+(n-1)\mu}e^{-H(t_2-t^{n-3})}e^{-(n-1)\mu(t_2-t)} + \frac{(n-1)\mu}{H+(n-1)\mu}e^{-H(t-t^{n-3})}$$
$$P_{n-2}'(t^{n-3}) = -\frac{H(n-1)\mu}{H+(n-1)\mu}[e^{-H(t-t^{n-3})} - e^{-H(t_2-t^{n-3})}e^{-(n-1)\mu(t_2-t)}] < 0.$$

Hence $P_{n-2}(t^{n-3})$ is decreasing in t. For $t^{n-3} > t$,

$$P_{n-2}(t^{n-3}) = e^{-H(t_2-t^{n-3})}e^{-(n-1)\mu(t_2-t^{n-3})} + \int_{t^{n-3}}^{t_2} e^{-H(s-t^{n-3})}(n-1)\mu e^{-(n-1)\mu(s-t^{n-3})}ds$$

is constant in t. Hence $V_{n-2}(t^{n-3})$ is decreasing t. By the induction above, the numerator of (28) is decreasing in t. Moreover, for the denominator, we have

$$\frac{\partial[\sum_{m=0}^{n-1} C_{n-1}^m(\int_{t_1}^t He^{-H(s-t_1)}e^{-\mu(t-s)}ds)^m(e^{-H(t-t_1)})^{n-m-1}]}{\partial t}$$
$$= \frac{\partial[(\frac{H}{H-\mu}e^{-\mu(t-t_1)} - \frac{\mu}{H-\mu}e^{-H(t-t_1)})^{n-1}]}{\partial t}.$$

Then the rest of the proof follows the proof of Theorem 1. Note that by A3(m) we have $\frac{1-\alpha}{\alpha}e^{nHt_1} > n-1$, thus the denominator of (28) is decreasing in t. In conclusion, (28) is decreasing in t for $t \in [t_1, t_2]$. As $(28) = c$ at $t = t_2$, for $t \in [t_1, t_2]$ the value of staying in the game is always larger than the cost, implying that the firm will not exit.

Part IV: prove that withholding the solution after solving stage 1 is optimal in the withhold region. At any $t \in [t_1, t_2)$, the expected payoff of disclosing is

$$\frac{\sum_{m=0}^{n-1} C_{n-1}^m(\int_{t_1}^t He^{-H(s-t_1)}e^{-\mu(t-s)}ds)^m(e^{-H(t-t_1)})^{n-m-1}(\frac{p_1}{m+1}+\frac{p_2}{n})}{\sum_{m=0}^{n-1} C_{n-1}^m(\int_{t_1}^t He^{-H(s-t_1)}e^{-\mu(t-s)}ds)^m(e^{-H(t-t_1)})^{n-m-1}}$$

Note that the above expression is a weighted average of $\frac{p_1}{m+1}+\frac{p_2}{n}$, $m = 1, 2, ..., n-1$, also note that $\int_{t_1}^t He^{-H(s-t_1)}e^{-\mu(t-s)}ds$ and $e^{-H(t-t_1)}$ are decreasing in t, and $\frac{p_1}{m+1}+\frac{p_2}{n}$ is decreasing in m. Then as t increases, the above expression puts heavier weights on smaller payoffs, and it's value is decreasing. Also, economically, it's straightforward that the the payoff of disclosing is decreasing in t, as time evolves and the other players are withholding, a player is less certain about his payoff in stage 1, thus disclosing is less attractive. The expected payoff of withholding the solution is $V(t)$, by Lemma 1, it is increasing in t. Then the difference between withholding and disclosing is increasing in t.

Part V: prove that disclosure is optimal after solving stage 1 in the disclose region. This part follows the proof of Theorem 1, with only minor adjustments to the payoffs.

Appendix H. Proof of Theorem 2

From our analysis in Section 4.1, the breakeven condition for a firm in a duopoly competition is

$$c = H\frac{(p_1+p_2)e^{-H(t_2-t_1)} + \frac{(p_1+p_2)}{2}\int_0^{t_2-t_1} He^{-Hs}e^{-(t_2-t_1-s)\mu}ds}{e^{-H(t_2-t_1)} + \int_0^{t_2-t_1} He^{-Hs}e^{-(t_2-t_1-s)\mu}ds + \frac{1-\alpha}{\alpha}e^{H(t_1+t_2)}}$$
$$= H\frac{(p_1+p_2)e^{-2Ht_2} + \frac{(p_1+p_2)}{2}e^{-H(t_1+t_2)}\int_0^{t_2-t_1} He^{-Hs}e^{-(t_2-t_1-s)\mu}ds}{e^{-2Ht_2} + e^{-H(t_1+t_2)}\int_0^{t_2-t_1} He^{-Hs}e^{-(t_2-t_1-s)\mu}ds + \frac{1-\alpha}{\alpha}}, \tag{29}$$

and the social welfare can be represented by

$$SW_{duo} = \lambda\int_0^{t_2} 2He^{-2Hs}(p^S-2sc)ds - (1-\lambda+\lambda e^{-2Ht_2})2t_2c$$
$$-\lambda\int_{t_1}^{t_2} 2He^{-2Hs}[\int_s^{t_2}(H+\mu)e^{-H(s'-s)}e^{-\mu(s'-s)}(s'-s)cds'$$
$$+e^{-H(t_2-s)}e^{-\mu(t_2-s)}(t_2-s)c]ds. \tag{30}$$

The first two terms represent the hypothetical social welfare if both firms start working on stage 2 once either firm solves stage 1; the third term represents the additional loss due to the existence of the withhold region, when it is possible that a firm still works on stage 1 while its opponent has completed the stage.

For a monopoly, let t_2' denote the exiting time. The breakeven condition is

$$c = H\frac{(p_1+p_2)e^{-Ht_2'}}{e^{-Ht_2'}+\frac{1-\alpha}{\alpha}}, \tag{31}$$

and the social welfare can be represented by

$$SW_{mono} = \lambda\int_0^{t_2'} He^{-Hs}(p^S-sc)ds - (1-\lambda+\lambda e^{-Ht_2'})t_2'c.$$

Note that t_2-t_1 is not affected by c, so as c changes, holding other parameters constant, the last term of the right-hand side of (30) is a constant multiplied by c. Suppose that $SW_{duo}=SW_{mono}$ at some cost level c'. Suppose that the cost increases from c' to c'' such that the exiting time decreases by $2k$ in the monopoly scenario. The part of hypothetical social welfare in the duopoly competition scenario would change by the same amount for the same c'', if the exiting time decreases by k. That is,

$$\lambda\int_0^{t_2-k} 2He^{-2Hs}(p^S-2sc'')ds - (1-\lambda+\lambda e^{-2Ht_2})2(t_2-k)c''$$
$$-\lambda\int_{t_1}^{t_2} 2He^{-2Hs}[\int_s^{t_2}(H+\mu)e^{-H(s'-s)}e^{-\mu(s'-s)}(s'-s)c'ds'$$
$$+e^{-H(t_2-s)}e^{-\mu(t_2-s)}(t_2-s)c']ds$$
$$= \lambda\int_0^{t_2'-2k} He^{-Hs}(p^S-sc'')ds - (1-\lambda+\lambda e^{-Ht_2'})(t_2'-2k)c''.$$

For an arbitrary cost level c, we rewrite the exiting time in competition as $t_2=t_2(c)$ and the beginning of the withhold region as $t_1=t_1(c)$, and the exiting time in monopoly as $t_2'=t_2'(c)$. The previous breakeven conditions at $c=c'$ can be written as

$$\begin{cases} c' = H\frac{(p_1+p_2)A+\frac{(p_1+p_2)}{2}B}{A+B+\frac{1-\alpha}{\alpha}} \\ c' = H\frac{(p_1+p_2)C}{C+\frac{1-\alpha}{\alpha}} \end{cases}$$

where $A=e^{-2Ht_2(c')}$, $B=e^{-H(t_1(c')+t_2(c'))}\int_0^{t_2(c')-t_1(c')}He^{-Hs}e^{-(t_2(c')-t_1(c')-s)\mu}ds$, $C=e^{-Ht_2'(c')}$. Notice that $\frac{(p_1+p_2)A+\frac{(p_1+p_2)}{2}B}{A+B} < \frac{(p_1+p_2)C}{C}$, thus $C<A+B$. Define x by equation $(p_1+p_2)A+\frac{(p_1+p_2)}{2}B = x(A+B)$, then we have $x<(p_1+p_2)$, $x(A+B) > (p_1+p_2)C$.

Suppose that at cost level c'' we have $2[t_2(c')-t_2(c'')] = 2[t_1(c')-t_1(c'')] = t_2'(c')-t_2'(c'')$. Let $y=e^{-H[t_2'(c'')-t_2'(c')]}$, and we have

$$\frac{(p_1+p_2)C}{C+\frac{1-\alpha}{\alpha}} = \frac{(p_1+p_2)A+\frac{(p_1+p_2)}{2}B}{A+B+\frac{1-\alpha}{\alpha}}$$
$$\Leftrightarrow (p_1+p_2)C[(A+B)y+\frac{1-\alpha}{\alpha}y] = x(A+B)(Cy+\frac{1-\alpha}{\alpha}y)$$

$$\Leftrightarrow (p_1+p_2)C[(A+B)y+\frac{1-\alpha}{\alpha}] > x(A+B)(Cy+\frac{1-\alpha}{\alpha})$$
$$\Leftrightarrow \frac{(p_1+p_2)Cy}{Cy+\frac{1-\alpha}{\alpha}} > \frac{x(A+B)y}{(A+B)y+\frac{1-\alpha}{\alpha}}$$

The second equivalence results from $x(A+B) > (p_1+p_2)C$. Then the right-hand side of (29) at $c = c''$, which is equal to $H\frac{x(A+B)y}{(A+B)y+\frac{1-\alpha}{\alpha}}$, would be smaller than the right-hand side of (31) at $c = c''$, which is equal to $H\frac{(p_1+p_2)Cy}{Cy+\frac{1-\alpha}{\alpha}}$. Thus we must have $2[t_2(c')-t_2(c'')] > t_2'(c')-t_2'(c'')$.

We use SW'_{duo} to denote the term

$$\lambda\int_0^{t_2(c'')} 2He^{-2Hs}(P^S-2sc'')ds-(1-\lambda+\lambda e^{-2Ht_2})2t_2(c'')c''$$
$$-\lambda\int_{t_1}^{t_2} 2He^{-2Hs}[\int_s^{t_2} He^{-H(s'-s)}e^{-\mu(s'-s)}(s'-s)c'ds'$$
$$+e^{-H(t_2-s)}e^{-\mu(t_2-s)}(t_2-s)c']ds$$
$$=\lambda\int_0^{t_2'(c)-2(t_2(c')-t_2(c''))} He^{-Hs}(p^S-sc'')ds$$
$$-(1-\lambda+\lambda e^{-Ht_2'})(t_2'(c)-2(t_2(c')-t_2(c'')))c''.$$

We know that $SW_{duo} < SW'_{duo} < SW_{mono}$. Similarly, we can show that when $c'' < c'$, $SW_{duo} > SW'_{duo} > SW_{mono}$. Hence, $c^* = c'$ is the desired cutoff value of cost.

Finally, suppose that $p^S = \infty$. In this case, the social welfare can be directly measured by the maximum total time spent in the game without solving stage 1.

From the breakeven conditions, we know that

$$2t_2(c)\begin{cases} > t_2'(c) \text{ if and only if } \frac{H(p_1+p_2)}{2} > c \\ = t_2'(c) \text{ if and only if } \frac{H(p_1+p_2)}{2} = c \\ < t_2'(c) \text{ if and only if } \frac{H(p_1+p_2)}{2} < c. \end{cases}$$

Hence, we can conclude that $c^* = \frac{H(p_1+p_2)}{2}$.

Appendix I. Proof of Theorem 3

Similar to the proof of Theorem 1, our proof has five parts. In Part I, we provide a set of necessary conditions to characterize a unique candidate for equilibrium. In Parts II-V, we show that these conditions are sufficient for an equilibrium.

Part I: characterize the equilibrium. Following the proof of Theorem 1, the firm that enters the withhold region first (assumed to be i here without loss of generality) has lower cost, i.e., $c^i < c^j$. Now, we have a set of necessary conditions for equilibrium cutoffs $t_1^i, t_1^j, t_2^j, t_2^i$: (1) $t_1^i = t_2^j - \frac{\ln\frac{H(p_1+p_2)}{p_1H-p_2\mu}}{H+\mu}$ if $t_2^j \geq \frac{\ln\frac{H(p_1+p_2)}{p_1H-p_2\mu}}{H+\mu}$, and $t_1^i = 0$ otherwise; (2) $t_1^j = t_2^i - \frac{\ln\frac{H(p_1+p_2)}{p_1H-p_2\mu}}{H+\mu}$ if $t_2^i - t_2^j \leq \frac{\ln\frac{H(p_1+p_2)}{p_1H-p_2\mu}}{H+\mu}$, and $t_1^j = t_2^j$ otherwise; (3) the condition for c^i in (5) if $t_2^i > 0$; and (4) the condition for c^j in (5) if $t_2^j > 0$. In the following steps of Part I, we show that for any $c^i > 0, c^j > c^i$, the cutoffs satisfying these conditions are unique.

Rewrite the condition for c^j in (5):

$$c^j = \frac{1}{2}H(p_1+p_2)[1+\frac{e^{-2Ht_2^j}(\frac{H}{H+\mu}e^{-(H+\mu)(t_2^i-t_2^j)}+\frac{\mu}{H+\mu})-\frac{1-\alpha}{\alpha}}{e^{-2Ht_2^j}+e^{-Ht_1^i-Ht_2^j}\int_0^{t_2^j-t_1^i}He^{-Hs}e^{-(t_2^j-t_1^i-s)\mu}ds+\frac{1-\alpha}{\alpha}}]. \tag{32}$$

When $t_1^i > 0$, by A3(a), the numerator is negative, and the first-order derivative with respect to t_1^i is

$$\frac{d[e^{-2Ht_2^j}(\frac{H}{H+\mu}e^{-(H+\mu)(t_2^i-t_2^j)}+\frac{\mu}{H+\mu})-\frac{1-\alpha}{\alpha}]}{dt_1^i}$$
$$=\frac{d[e^{-2Ht_2^j}(\frac{H}{H+\mu}e^{-(H+\mu)(t_2^i-t_2^j)}+\frac{\mu}{H+\mu})]}{dt_2^j}$$
$$=\frac{-H^2+H\mu}{H+\mu}e^{-(H+\mu)t_2^i-(H-\mu)t_2^j}+\frac{-2H\mu}{H+\mu}e^{-2Ht_2^j})$$
$$<0.$$

Hence, the numerator decreases in t_1^i. The denominator is positive and decreases in t_1^i. Thus, the right-hand side of (32) decreases in t_1^i with limit 0 as $t_1^i \to \infty$.

When $t_1^i = 0,\ t_2^j \le \frac{H(p_1+p_2)}{H+\mu}$, and (32) becomes

$$c_j = \frac{(p_1+p_2)}{2}[1 + \frac{e^{-2Ht_2^j}(1 - \int_0^{t_2^i - t_2^j} He^{-Hs}e^{-\mu s}ds) - \frac{1-\alpha}{\alpha}}{e^{-2Ht_2^j} + e^{-Ht_2^j}\int_0^{t_2^j} He^{-Hs}e^{-(t_2^j - s)\mu}ds + \frac{1-\alpha}{\alpha}}]$$
$$= \frac{(p_1+p_2)}{2}[1 + \frac{e^{-2Ht_2^j}(\frac{H}{H+\mu}e^{-(H+\mu)(t_2^i - t_2^j)} + \frac{\mu}{H+\mu}) - \frac{1-\alpha}{\alpha}}{\frac{H}{H-\mu}e^{-(H+\mu)t_2^j} - \frac{\mu}{H-\mu}e^{-2Ht_2^j} + \frac{1-\alpha}{\alpha}}].$$

By the above argument, the numerator is negative and decreases in t_2^j. The derivative of the denominator with respect to t_2^j is $\frac{-H^2 - H\mu}{H-\mu}e^{-(H+\mu)t_2^j} + \frac{2H\mu}{H-\mu}e^{-2Ht_2^j}$, which is negative. Hence, the denominator is positive and increases in t_2^j. Thus, the right-hand side of (32) decreases in t_2^j.

Finally, when $t_2^j = 0,\ c^j$ can be any number larger than the right-hand side of (32) to make firm j's instantaneous net payoff rate negative at $t = 0$. In this case, we let $t_2^j = 0$. By the continuity of the right-hand side of the above expressions, for every $c^j > 0$, we can now identify unique $t_1^i,\ t_2^j$ such that conditions (1) - (4) are satisfied.

Next, we fix t_1^i and t_2^j according to c^j and rewrite the condition for c^i in (5):

$$c_i = \frac{e^{-Ht_2^j - Ht_2^i}(p_1+p_2) + e^{-Ht_1^j - Ht_2^i}\frac{(p_1+p_2)}{2}\int_0^{t_2^j - t_1^j} He^{-Hs}e^{-(t_2^i - t_1^j - s)\mu}ds}{e^{-Ht_2^j - Ht_2^i} + e^{-Ht_1^j - Ht_2^i}\int_0^{t_2^j - t_1^j} He^{-Hs}e^{-(t_2^i - t_1^j - s)\mu}ds + \frac{1-\alpha}{\alpha}}$$
$$= \frac{(p_1+p_2)}{2}[1 + \frac{e^{-Ht_2^j - Ht_2^i} - \frac{1-\alpha}{\alpha}}{e^{-Ht_2^j - Ht_2^i} + e^{-Ht_1^j - Ht_2^i}\int_0^{t_2^j - t_1^j} He^{-Hs}e^{-(t_2^i - t_1^j - s)\mu}ds + \frac{1-\alpha}{\alpha}}].$$

If $t_1^j = t_2^j$ and $t_2^i - t_1^j \ge \frac{\ln\frac{H(p_1+p_2)}{p_1H - p_2\mu}}{H+\mu}$, we have

$$c^i = \frac{(p_1+p_2)}{2}[1 + \frac{e^{-Ht_2^j - Ht_2^i} - \frac{1-\alpha}{\alpha}}{e^{-Ht_2^j - Ht_2^i} + \frac{1-\alpha}{\alpha}}].$$

By A3(a), the numerator is negative and decreases in t_2^i, and the denominator is positive and decreases in t_2^i. Thus, the right-hand side decreases in t_2^i with limit 0 as $t_2^i \to \infty$. If $t_1^i < t_1^j < t_2^j$ and $t_2^i - t_1^j = \nabla = \frac{\ln\frac{H(p_1+p_2)}{p_1H - p_2\mu}}{H+\mu}$, we have

$$c^i = \frac{(p_1+p_2)}{2}[1 + \frac{e^{-Ht_2^j - Ht_2^i} - \frac{1-\alpha}{\alpha}}{e^{-Ht_2^j - Ht_2^i} + e^{-Ht_1^j - Ht_2^i}\int_0^{t_2^j - t_1^j} He^{-Hs}e^{-(\nabla - s)\mu}ds + \frac{1-\alpha}{\alpha}}].$$

The numerator is negative and decreases in t_2^i, and the denominator is positive and decreases in t_2^i. Thus, the right-hand side decreases in t_2^i. If $t_1^j = t_1^i = 0$ and $t_2^j < t_2^i \le \frac{\ln\frac{H(p_1+p_2)}{p_1H - p_2\mu}}{H+\mu}$, we have

$$c^i = \frac{(p_1+p_2)}{2}[1 + \frac{e^{-Ht_2^j - Ht_2^i} - \frac{1-\alpha}{\alpha}}{e^{-Ht_2^j - Ht_2^i} + e^{-Ht_1^j - Ht_2^i}\int_0^{t_2^j} He^{-Hs}e^{-(t_2^i - s)\mu}ds + \frac{1-\alpha}{\alpha}}].$$

The numerator is negative and decreases in t_2^i, and the denominator is positive and decreases in t_2^i. Thus, the right-hand side decreases in t_2^i. Finally, if $t_1^j = t_1^i > 0$ or if $t_1^j = t_1^i = 0$ and $t_2^j = t_2^i$, we simply have $c^i = c^j$. By the continuity of the right-hand side of the above expressions, for every $c^i < c^j$, t_1^i and t_2^j, we can now identify unique t_2^i and t_1^j such that conditions (1) - (4) are satisfied. This completes Part I.

In Parts II-V, we prove that every set of cutoffs satisfying these conditions indeed constitutes an equilibrium.

Part II: prove that firm $i,\ j$ will never remain in the game after $t_2^i,\ t_2^j$,**respectively.** Note that t_2^i is the last cutoff; firm j exits before t_2^i. Thus, for firm i, we refer to the proof of Theorem 1, Part II. The difference is that we impose assumption A3(a) instead and replace $t_1,\ t_2$ with $t_1^j,\ t_2^j$, respectively. Following the proof, after t_2^j, firm i's incentive for remaining in the game decreases in time. Since $t_2^i > t_2^j$, the firm's incentive to remain also decreases in time after t_2^i. The remainder of the proof can be readily applied.

For firm j, its opponent is withholding the solution in $[t_1^i, t_2^j]$. For any alternative exit time $t_2^{j\prime} \in [t_2^j, t_2^i]$, at $t_2^{j\prime}$, the instantaneous payoff rate from remaining in the game is

$$\tilde{\lambda}^j(t_2^{j'})H\frac{e^{-H(t_2^{j'}-t_1^i)}[(\frac{\mu}{H+\mu}+\frac{H}{H+\mu}e^{-(H+\mu)(t_2^i-t_2^{j'})})(p_1+p_2)+\frac{H}{H+\mu}(1-e^{-(H+\mu)(t_2^i-t_2^{j'})})\frac{p_1+p_2}{2}]+\frac{(p_1+p_2)}{2}\int_0^{t_2^{j'}-t_1^i}He^{-Hs}e^{-(t_2^{j'}-t_1^i-s)\mu}ds}{e^{-H(t_2^{j'}-t_1^i)}+\int_0^{t_2^{j'}-t_1^i}He^{-Hs}e^{-(t_2^{j'}-t_1^i-s)\mu}ds}$$
$$=H\frac{p_1+p_2}{2}[1+\frac{e^{-2Ht_2^{j'}}(\frac{H}{H+\mu}e^{-(H+\mu)(t_2^i-t_2^{j'})}+\frac{\mu}{H+\mu})-\frac{1-\alpha}{\alpha}}{e^{-Ht_1^i-Ht_2^{j'}}\int_{t_1^i}^{t_2^{j'}}He^{-H(s-t_1^i)}e^{-\mu(t_2^{j'}-s)}ds+e^{-2Ht_2^{j'}}+\frac{1-\alpha}{\alpha}}]$$

By A3(a), the numerator is negative, and its first-order derivative with respect to $t_2^{j'}$ is

$$\frac{d[e^{-2Ht_2^{j'}}(\frac{H}{H+\mu}e^{-(H+\mu)(t_2^i-t_2^{j'})}+\frac{\mu}{H+\mu})-\frac{1-\alpha}{\alpha}]}{dt_2^{j'}}$$
$$=\frac{d[e^{-2Ht_2^{j'}}(\frac{H}{H+\mu}e^{-(H+\mu)(t_2^i-t_2^{j'})}+\frac{\mu}{H+\mu})]}{dt_2^{j'}}$$
$$=\frac{-H^2+H\mu}{H+\mu}e^{-(H+\mu)t_2^i-(H-\mu)t_2^{j'}}+\frac{-2H\mu}{H+\mu}e^{-2Ht_2^{j'}})$$
$$<0.$$

Thus, the numerator decreases in $t_2^{j'}$. The denominator is positive, and its derivative with respect to $t_2^{j'}$ is

$$\frac{d[e^{-Ht_1^i-Ht_2^{j'}}\int_{t_1^i}^{t_2^{j'}}He^{-H(s-t_1^i)}e^{-\mu(t_2^{j'}-s)}ds+e^{-2Ht_2^{j'}}+\frac{1-\alpha}{\alpha}]}{dt_2^{j'}}$$
$$=\frac{d[\frac{H}{H-\mu}e^{-(H-\mu)t_1^i-(H+\mu)t_2^{j'}}-\frac{\mu}{H-\mu}e^{-2Ht_2^{j'}}+\frac{1-\alpha}{\alpha}]}{dt_2^{j'}}$$
$$=\frac{-H^2-H\mu}{H-\mu}e^{-(H-\mu)t_1^i-(H+\mu)t_2^{j'}}+\frac{2H\mu}{H-\mu}e^{-2Ht_2^{j'}}$$
$$<0.$$

Thus, the denominator decreases in $t_2^{j'}$. We can then conclude that the instant payoff rate is decreasing in $t_2^{j'}$. As the breakeven condition holds at t_2^j, every $t_2^{j'}$ larger than t_2^j renders a negative instantaneous net payoff rate from $t_2^{j'}$ onwards and thus is dominated by exiting at t_2^j.

For any alternative exit time $t_2^{j'}>t_2^i$, since $t_2^{j'}$ becomes the last cutoff, we can follow our proof for firm i.

Part III: prove that firm i, j will never exit before t_2^i, t_2^j,**respectively.** We refer to the proof of Theorem 1, Part III with the following adjustment.

For firm j, suppose that at $t_2^{j'}\in[t_1^j,t_2^j]$, it exits without solving stage 1 and withholds the solution otherwise (disclosure is not optimal after $=t_1^j$, see Part IV[7]). By Part II, the instantaneous payoff rate before exiting decreases in $t_2^{j'}$ and thus is larger than c^j. Exiting at $t_2^{j'}$ cannot be a preferred deviation.

For firm j, suppose that at $t_2^{j'}\in[t_1^i,t_1^j]$, it exits without solving stage 1 and discloses the solution otherwise (withholding the solution is not optimal before t_1^j, see Part V). The instantaneous payoff rate at $t_2^{j'}$ is

$$\frac{\tilde{\lambda}(t_2^{j'})H[(p_1+\frac{p_2}{2})e^{-H(t_2^{j'}-t_1^i)}+\frac{p_1+p_2}{2}\int_{t_1^i}^{t_2^{j'}}He^{-H(s-t_1^i)}e^{-\mu(t_2^{j'}-s)}ds]}{e^{-H(t_2^{j'}-t_1^i)}+\int_{t_1^i}^{t_2^{j'}}He^{-H(s-t_1^i)}e^{-\mu(t_2^{j'}-s)}ds}$$
$$=\frac{H[(p_1+\frac{p_2}{2})e^{-H(t_2^{j'}-t_1^i)}+\frac{p_1+p_2}{2}\int_{t_1^i}^{t_2^{j'}}He^{-H(s-t_1^i)}e^{-\mu(t_2^{j'}-s)}ds]}{e^{-H(t_2^{j'}-t_1^i)}+\int_{t_1^i}^{t_2^{j'}}He^{-H(s-t_1^i)}e^{-\mu(t_2^{j'}-s)}ds+\frac{1-\alpha}{\alpha}e^{Ht_1^i+Ht_2^{j'}}}$$
$$=H\frac{p_1+p_2}{2}[1+\frac{\frac{p_1}{p_1+p_2}e^{-2Ht_2^{j'}}-\frac{1-\alpha}{\alpha}}{e^{-2Ht_2^{j'}}+e^{-Ht_1^i-Ht_2^{j'}}\int_{t_1^i}^{t_2^{j'}}He^{-H(s-t_1^i)}e^{-\mu(t_2^{j'}-s)}ds+\frac{1-\alpha}{\alpha}}].$$

[7] Parts IV and V do not rely on the breakeven conditions, and thus, we can regard the t_2^i and t_2^j in the proof of Parts IV and V as arbitrarily given parameters.

The numerator is negative and decreases in $t_2^{j\prime}$. By Part II, the denominator is positive and decreases in $t_2^{j\prime}$. Hence, the instantaneous payoff rate is weakly larger than that at t_1^j, which is larger than c^j by the characterization of conditions (1) - (4).

For firm j, suppose that at $t_2^{j\prime} \in [0, t_1^i]$, it exits without solving stage 1 and discloses the solution otherwise. We refer to the proof of Theorem 1, Part III. The instantaneous payoff rate from remaining for another dt and disclosing any incoming solution is greater than c^j.

For firm i, suppose that at $t_2^{i\prime} \in [t_2^j, t_2^i]$, it exits without solving stage 1 and withholds the solution otherwise (disclosure is not optimal after t_1^i, see Part IV). By Part II, after t_2^j, firm i's incentive to remain in the game decreases over time. As the breakeven condition holds at t_2^i, any $t_2^{i\prime} \in [t_2^j, t_2^i]$ renders a positive instantaneous net payoff rate at $t_2^{i\prime}$ and thus cannot be a preferred deviation.

For firm i, suppose that at $t_2^{i\prime} \in [t_1^j, t_2^j]$, it exits without solving stage 1 and withholds the solution otherwise. We refer to the proof of Theorem 1, Part III. The difference is that we replace t_1, t_2 with t_1^j, $t_2^{i\prime}$, respectively.

For firm i, suppose that at $t_2^{i\prime} \in [0, t_1^j]$, it exits without solving stage 1 and discloses the solution otherwise (withholding the solution is not optimal before t_1^i, see Part V). We refer to the proof of Theorem 1, Part III. The instantaneous payoff rate from remaining for another dt and disclosing any incoming solution is greater than c^i.

Part IV: prove that withholding the solution after solving stage 1 is optimal in the withhold region. We separately discuss the incentives of the two firms.

For firm j, we refer to the proof of Theorem 1, Part IV, by using t_1^i, t_2^i instead of t_1, t_2.

For firm i, by our assumption, firm i will not disclose the solution after t_2^j. For $t \in [t_1^j, t_2^j]$, we refer to the proof of Theorem 1, Part IV, and use t_1^j, t_2^j instead of t_1, t_2. For $t \in [t_1^i, t_1^j]$, the expected payoff from disclosure is $p_1 + \frac{p_2}{2}$; the expected payoff from withholding the solution is

$$e^{-H(t_1^j - t)} e^{-\mu(t_1^j - t)} (p_1 + \frac{p_2}{2}) + \frac{(p_1 + p_2)}{2} \int_t^{t_1^j} H e^{-H(s-t)} e^{-\mu(s-t)} ds$$
$$+ (p_1 + p_2) \int_t^{t_1^j} \mu e^{-H(s-t)} e^{-\mu(s-t)} ds.$$

The difference is

$$\begin{aligned} & p_1 + \frac{p_2}{2} - e^{-H(t_1^j - t)} e^{-\mu(t_1^j - t)} (p_1 + \frac{p_2}{2}) \\ & - \frac{(p_1 + p_2)}{2} \int_t^{t_1^j} H e^{-H(s-t)} e^{-\mu(s-t)} ds - (p_1 + p_2) \int_t^{t_1^j} \mu e^{-H(s-t)} e^{-\mu(s-t)} ds \\ = & \ (p_2 \mu - p_1 H) \frac{1}{2} \int_t^{t_1^j} e^{-H(s-t)} e^{-\mu(s-t)} ds < 0. \end{aligned}$$

Thus, withholding the solution is preferred to disclosure.

Part V: prove that disclosure is optimal after solving stage 1 in the disclose region. We separately discuss the incentives of the two firms.

For firm i, we refer to the proof of Theorem 1, Part V. We use t_1^i instead of t_1.

For firm j, the instantaneous payoff rate from disclosure at $t \in [0, t_1^j]$ is

$$\begin{aligned} & \frac{\tilde{\lambda}(t) H[(p_1 + \frac{p_2}{2}) e^{-H(t - t_1^i)} + \frac{p_2}{2} \int_{t_1^i}^t H e^{-H(s - t_1^i)} e^{-\mu(t-s)} ds]}{e^{-H(t - t_1^i)} + \int_{t_1^i}^t H e^{-H(s - t_1^i)} e^{-\mu(t-s)} ds} \\ = & \frac{H[(p_1 + \frac{p_2}{2}) e^{-H(t - t_1^i)} + \frac{p_2}{2} \int_{t_1^i}^t H e^{-H(s - t_1^i)} e^{-\mu(t-s)} ds]}{e^{-H(t - t_1^i)} + \int_{t_1^i}^t H e^{-H(s - t_1^i)} e^{-\mu(t-s)} ds + \frac{1-\alpha}{\alpha} e^{H t_1^i + H t}}. \end{aligned}$$

By the proof of Lemma 3, there is no profitable type-1 deviation. The remainder of the proof follows that of Theorem 1, Part IV (if firm j deviates to withholding the solution after $t_1^{j\prime} \in [t_1^i, t_1^j]$, noting that the difference between withholding and disclosing the solution increases in t when the opponent is withholding the solution), and our above analysis for firm i (if firm j deviates to withholding the solution after $t_1^{j\prime} \in [0, t_1^i]$).

Appendix J. Figures

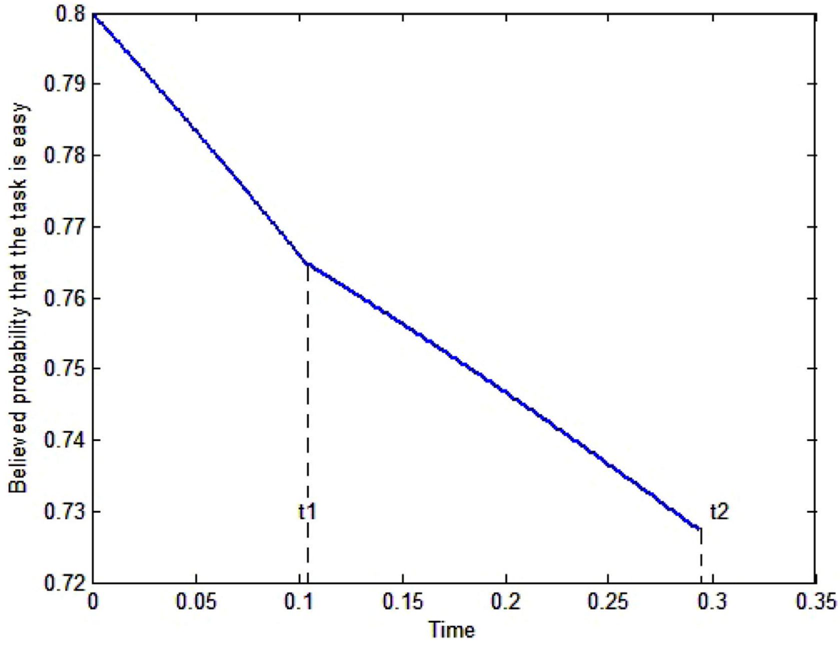

Fig. 2. Trajectory of belief.

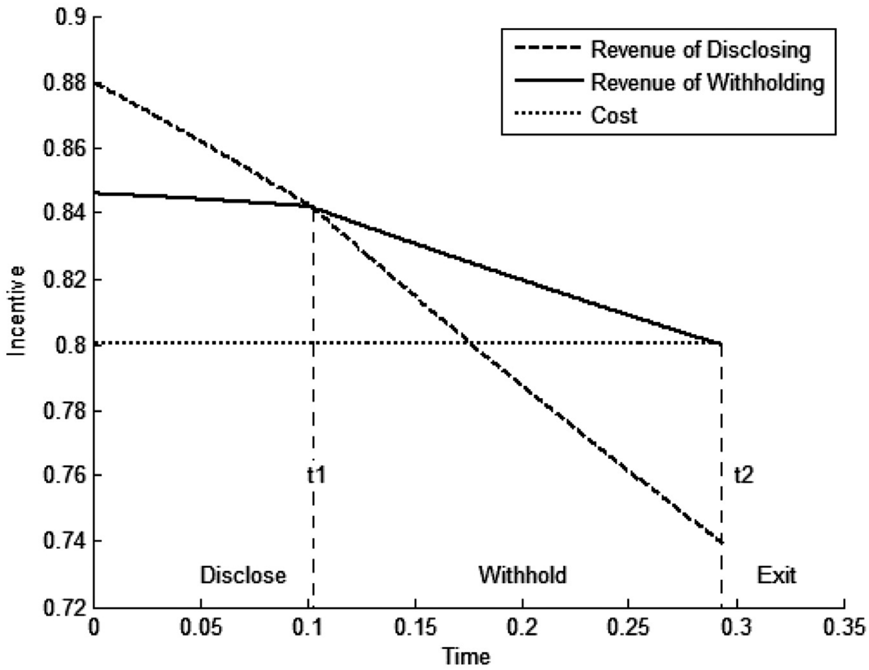

Fig. 3. Trajectory of cost and expected benefit.

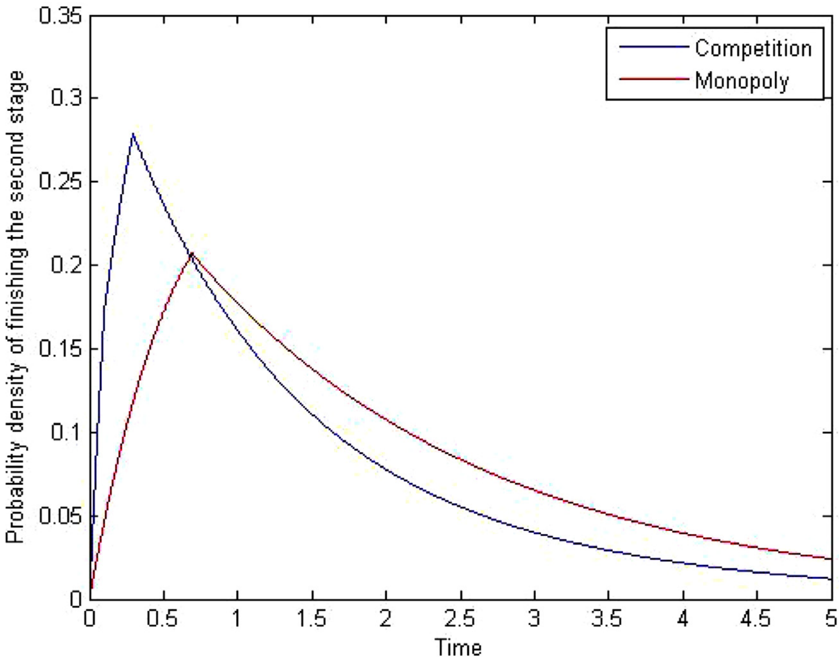

Fig. 4. Pdfs of whole project's success.

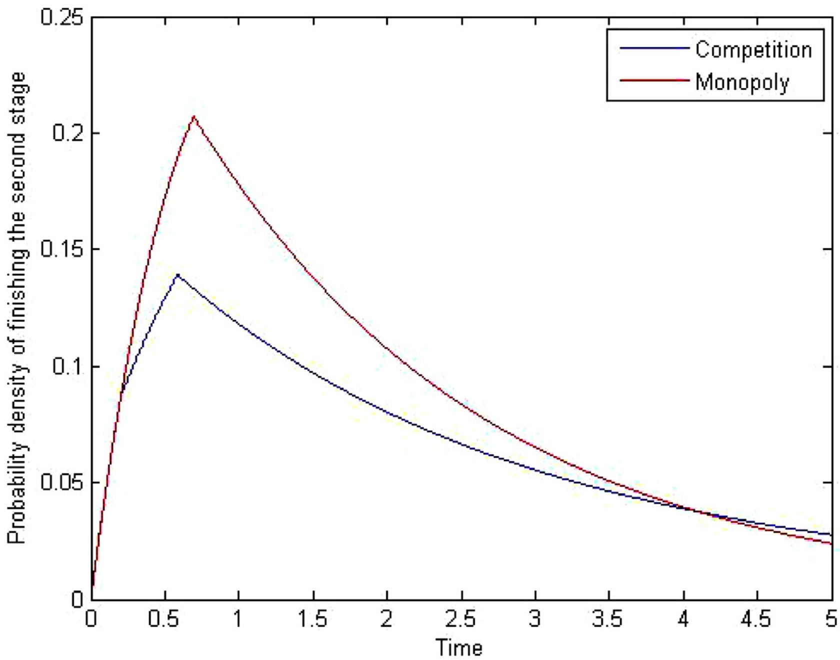

Fig. 5. Pdfs of whole project's success, with pdf under competition rescaled.

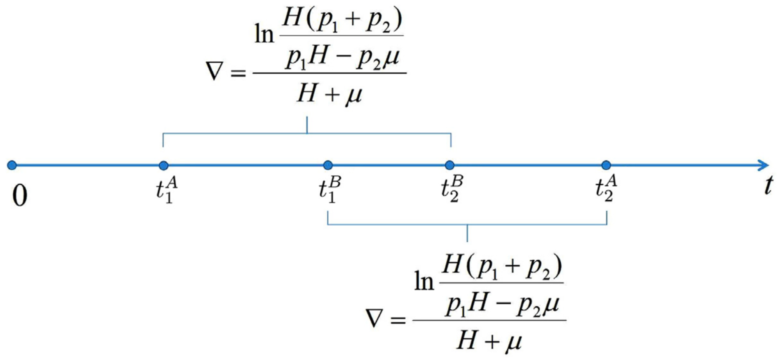

Fig. 6. Typical equilibrium with asymmetric costs.

References

Akcigit, U., Liu, Q., 2016. The role of information in innovation and competition. J. Eur. Econ. Assoc. 14 (4), 828–870.
Anton, J.J., Yao, D.A., 2003. Patents, invalidity, and the strategic transmission of enabling information. J. Econ. Manag. Strat. 12 (2), 151–178.
Bag, P.K., Dasgupta, S., 1995. Strategic r&d success announcements. Econ. Lett. 47, 17–26.
Baker, S., Mezzetti, C., 2005. Disclosure as a strategy in the patent race. J. Law Econ. 48 (1), 173–194.
Bhattacharya, S., Ritter, J.R., 1983. Innovation and communication: signalling with partial disclosure. Rev. Econ. Stud. 50 (2), 331–346.
Bloch, F., Markowitz, P., 1996. Optimal disclosure delay in multistage R&D competition. Int. J. Ind. Organ. 14 (2), 159–179.
Bobtcheff, C., Bolte, J., Mariotti, T., 2017. Researcher's dilemma. Rev. Econ. Stud. 84 (3), 969–1014.
Chatterjee, K., Evans, R., 2004. Rivals' search for buried treasure: competition and duplication in. RAND J. Econ. 35 (1), 160–183.
Choi, J.P., 1991. Dynamic R&D competition under "hazard rate" uncertainty. RAND J. Econ. 22 (4), 596–610.
Das, K., Klein, N., 2020. Over-and under-experimentation in a patent race with private learning. Working paper.
Das, K., Klein, N., Schmid, K., 2020. Strategic experimentation with asymmetric players. Econ. Theory 69 (4), 1147–1175.
Dasgupta, P., Stiglitz, J., 1980. Industrial structure and the nature of innovative activity. Econ. J. 90 (358), 266–293.
Dasgupta, P., Stiglitz, J., 1980. Uncertainty, industrial structure, and the speed of R&D. Bell J. Econ. 11 (1), 1–28.
De Fraja, G., 1993. Strategic spillovers in patent races. Int. J. Ind. Organ. 11 (1), 139–146.
Denicolo, V., Franzoni, L.A., 2004. Patents, secrets, and the first-inventor defense. J. Econ. Manag. Strat. 13 (3), 517–538.
Dong, M., 2018. Strategic experimentation with asymmetric information. Working paper.
Gill, D., 2008. Strategic disclosure of intermediate research results. J. Econ. Manag. Strat 17 (3), 733–758.
Grossman, G.M., Shapiro, C., 1987. Dynamic R&D competition. Econ. J. 97 (386), 372–387.
Guo, Y., Roesler, A.-K., 2018. Private learning and exit decisions in collaboration. Working paper.
Harris, C., Vickers, J., 1987. Racing with uncertainty. Rev. Econ. Stud. 54 (1), 1–21.
Heidhues, P., Rady, S., Strack, P., 2015. Strategic experimentation with private payoffs. J. Econ. Theory 159, 531–551.
Hendricks, K., 1992. Reputations in the adoption of a new technology. Int. J. Ind. Organ. 10 (4), 663–677.
Hopenhayn, H.A., Squintani, F., 2016. Patent rights and innovation disclosure. Rev. Econ. Stud. 83 (1), 199–230.
Horner, J., Skrzypacz, A., 2017. Learning, experimentation and information design. In: Honore, B., Pakes, A., Piazzesi, M., Samuelson, L. (Eds.), Advances in Economics and Econometrics: Volume 1: Eleventh World Congress. Cambridge University Press, Cambridge, pp. 63–97. chapter 2
Keller, G., Rady, S., 2010. Strategic experimentation with poisson bandits. Theore. Econ. 5 (2), 275–311.
Keller, G., Rady, S., Cripps, M., 2005. Strategic experimentation with exponential bandits. Econometrica 73 (1), 39–68.
Kultti, K., Takalo, T., Toikka, J., 2007. Secrecy versus patenting. RAND J. Econ. 38 (1), 22–42.
Lee, T., Wilde, L.L., 1980. Market structure and innovation: A reformulation. Q. J. Econ. 94 (2), 429–436.
Lichtman, D., Baker, S., Kraus, K., 2000. Strategic disclosure in the patent system. Vanderbilt Law Rev. 53 (6), 2175–2217.
Loury, G.C., 1979. Market structure and innovation. Q. J. Econ. 93 (3), 395–410.
Malueg, D.A., Tsutsui, S.O., 1996. Duopoly information exchange: the case of unknown slope. Int. J. Ind. Organ. 14, 119–136.
Moscarini, G., Squintani, F., 2010. Competitive experimentation with private information: the survivor's curse. J. Econ. Theory 145, 639–660.
Parchomovsky, G., 2000. Publish or perish. Michigan Law Rev. 98, 926–952.
Reinganum, J.F., 1981. Dynamic games of innovation. J. Econ. Theory 25 (1), 21–41.
Scotchmer, S., Green, J., 1990. Novelty and disclosure in patent law. RAND J. Econ. 21 (1), 131–146.